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early mathematics education
INNOVATIONS

Summer Institute, Day 4
6/21/2012

Agenda

- *Begin with Breakfast*
- Investigation: Fractions in the World
  - What is a whole? What is an equal part?
- Video Commentary
- CCSS for Mathematical Practice
  - What do they mean? How do they look & sound?
- Math Games: Mancala & 21
- *Break for Lunch*
- Effective Strategies for Teaching Math
  - How do they work?
- *Gathering to End the Day*

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How can we share fairly?

An important issue for children of all ages

Draw pictures to help you solve these brownie-sharing tasks.

Suppose 4 children are sharing 10 brownies so that each child will get the same amount. How much can each child have?

What about 5 brownies shared between 2 children?

What about 2 brownies shared among 4 children?

What about 5 brownies shared among 4 children?

What about 4 brownies shared among 8 children?

What about 3 brownies shared among 4 children?

Did everyone at your table draw their solutions the same way?

Which of the brownie-sharing tasks is most difficult? least difficult? Which tasks are the same degree of difficulty as each other?

Draw pictures to help you solve these pizza-sharing tasks. Can you come up with more than one way to show the same sharing task?

Suppose 6 children are sharing 4 pizzas so that each child will get the same amount. How much can each child have?

What about 7 pizzas shared among 6 children?

What about 5 pizzas shared among 3 children?

Were the pizza-sharing problems easier or harder than the brownie-sharing ones? Why?

Draw pictures to help you solve these cookie-sharing tasks.

Suppose 3 children are sharing 2 dozen cookies so that each child will get the same amount. How many cookies can each child have? What portion of the cookies does each child get?

What about 2 dozen cookies shared among 4 children?

What about 2 dozen cookies shared among 12 children?

What about 30 cookies shared among 3 children?

What about 30 cookies shared among 5 children?

Did you draw different kinds of pictures for the cookie-sharing tasks than for the pizza-sharing & brownie-sharing ones? Why?

Were the cookie-sharing problems easier or harder than the pizza-sharing or brownie-sharing ones? Why?

Draw a picture to help you solve this running-sharing task.

Suppose 4 children are going to run a 3-mile relay race, dividing it up so that each child runs the same amount. How far would each child need to run?

Did you draw a different kind of picture for this running-sharing task than for the cookie-sharing, pizza-sharing & brownie-sharing ones? Why?

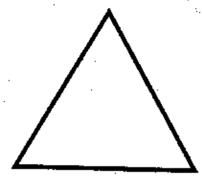
Was this running-sharing task easier or harder than the cookie-sharing, pizza-sharing or brownie-sharing ones? Why?

Based on discussion in Van de Walle, J.A. & Lovin, L.H. *Teaching Student-Centered Mathematics, Grades K-3.* Allyn and Bacon, 2006.

What are the Big Ideas about Fractions?

Topic	Big Ideas	Examples
Fractions (Parts & Wholes)	•Fractions are equal parts of a whole.	•Pizzas can be cut into 6 equal wedge-shaped pieces. •6 cars can be divided equally by giving 2 cars each to 3 children.
	•A whole or unit can be divided into equal parts in many different ways.	•A pizza can be divided into 4 or 6 or 8 equal slices. •6 cars can be divided equally into 2 groups of 3, 3 groups of 2, or 6 groups of 1.
	•A unit may be a single object or may be a collection of things.	•One whole pizza can be divided into equal slices. •In one group of 6 cars, 1/3 are red: 2 cars are red.

Major support for the Early Mathematics Education Project is provided by the McCormick Foundation, CME Group Foundation, the U.S. Dept. of Education Investing in Innovation Fund (i3), Motorola Foundation, Chicago Public Schools Office of Early Childhood Education, Exelon Corporation, and the Robert and Isabelle Bass Foundation



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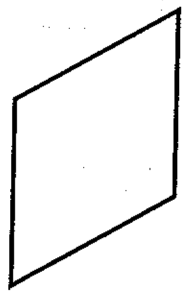

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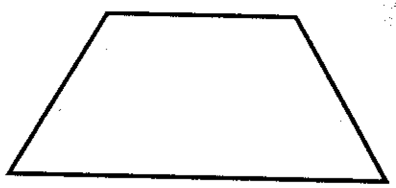

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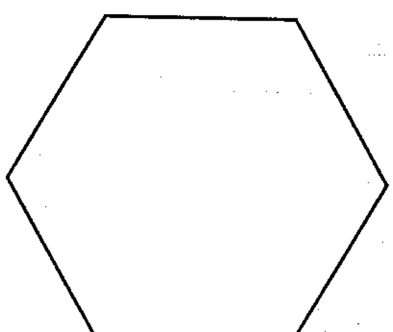

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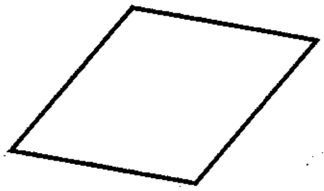

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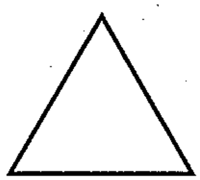

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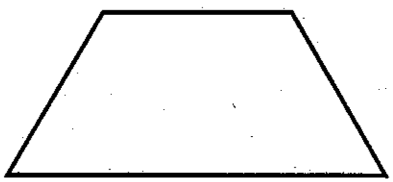

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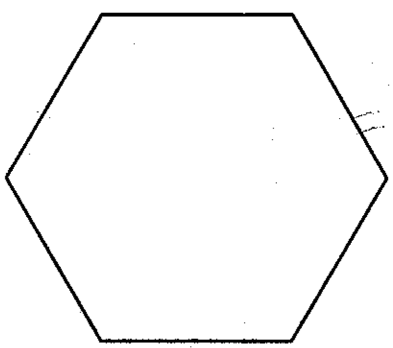

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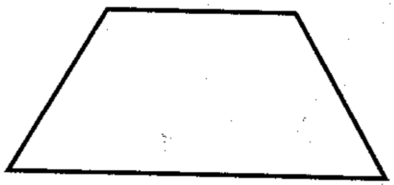

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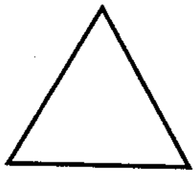

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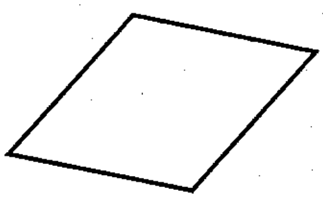

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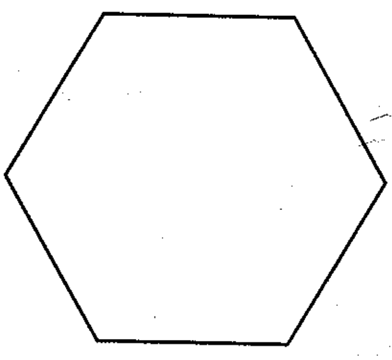

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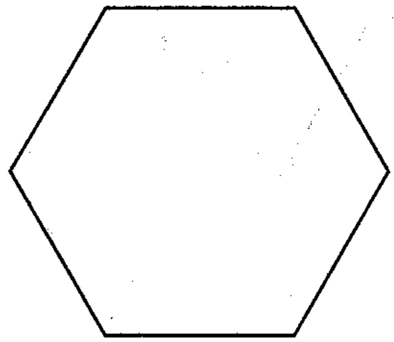
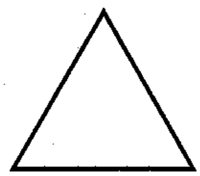
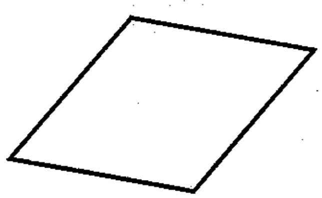
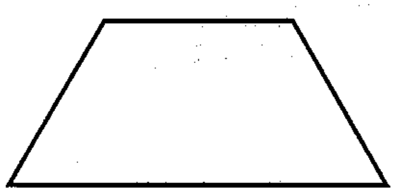
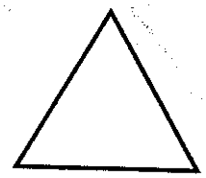
 = 1

 = ____

 = ____

 = ____



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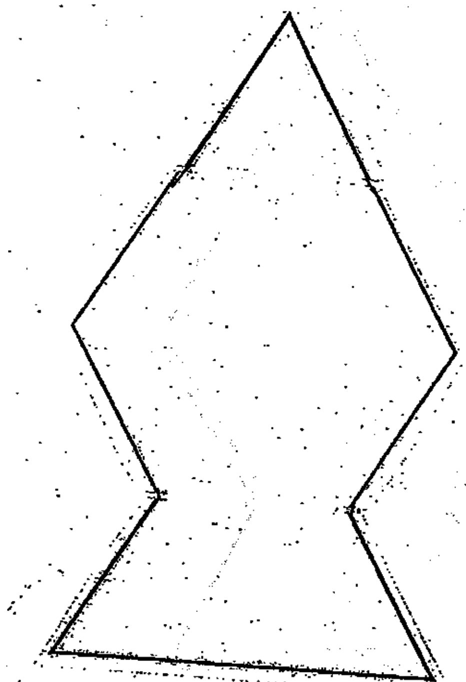

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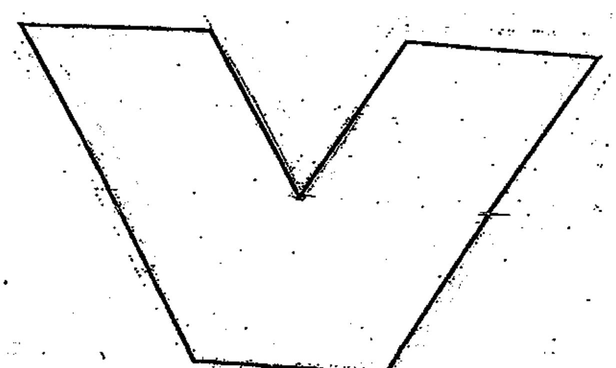

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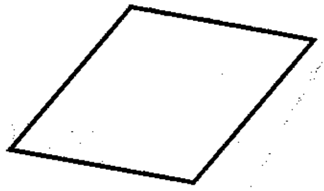

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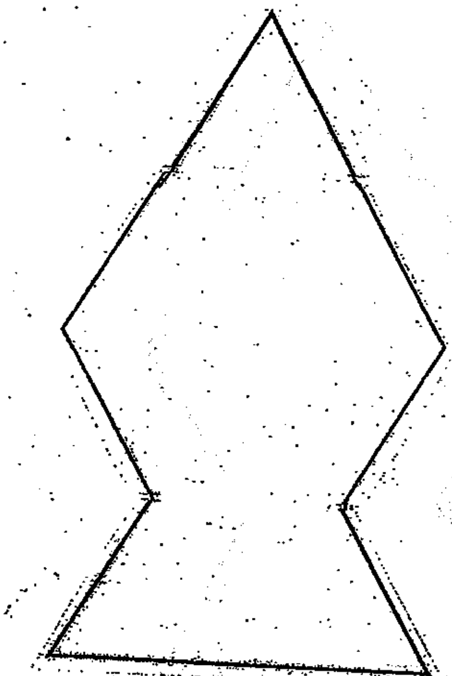

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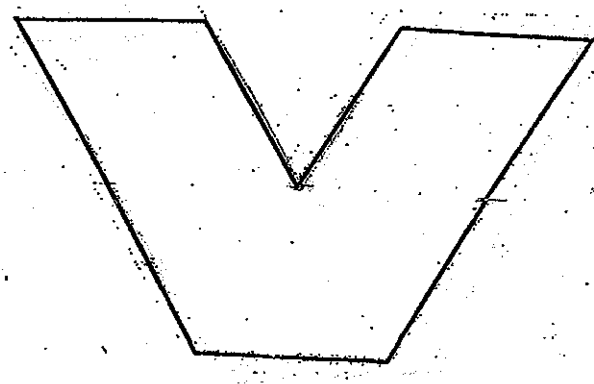

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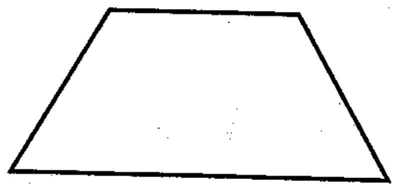

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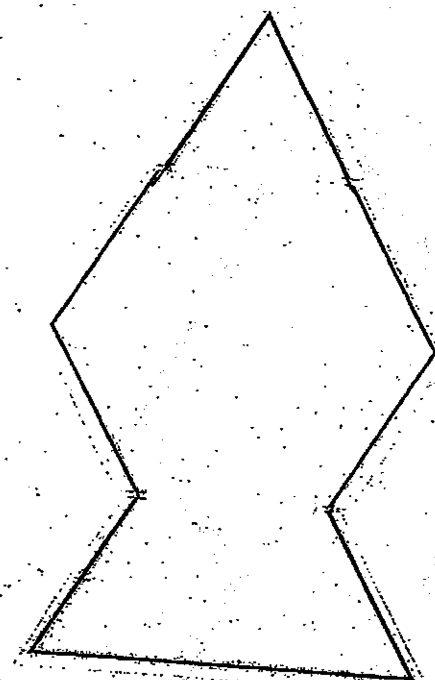

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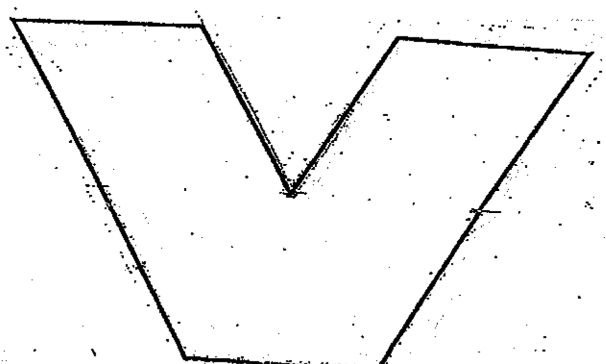

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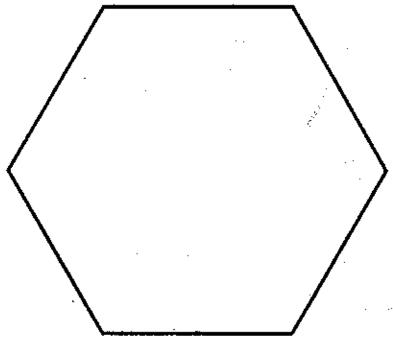

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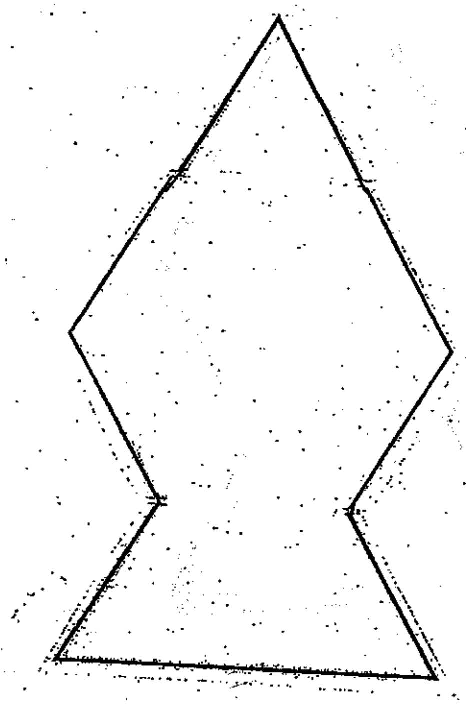

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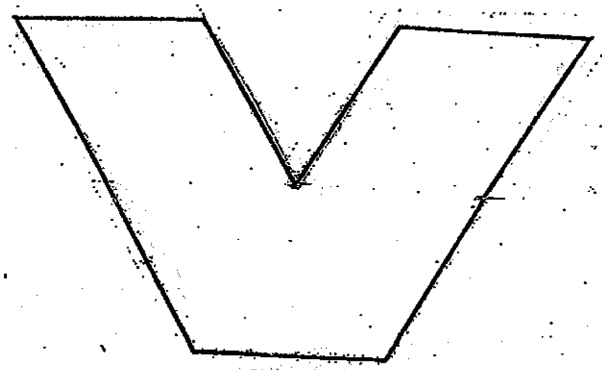

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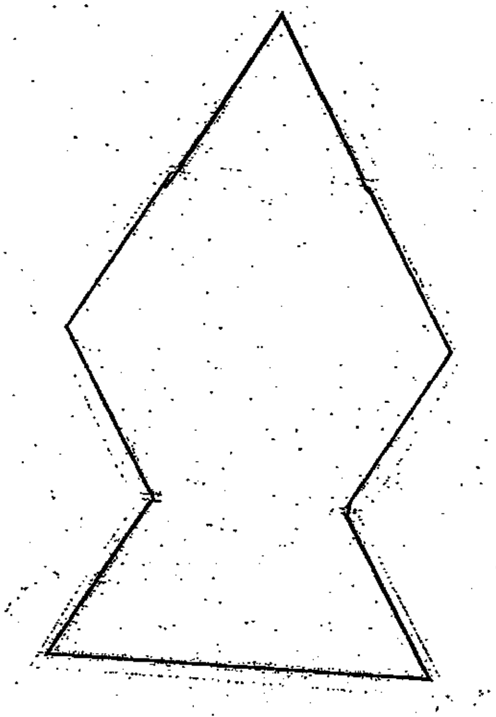

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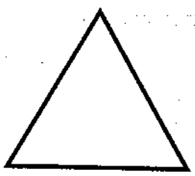

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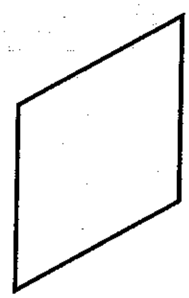

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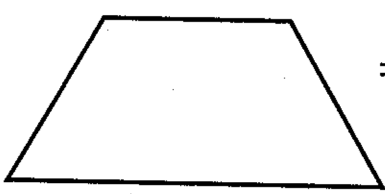

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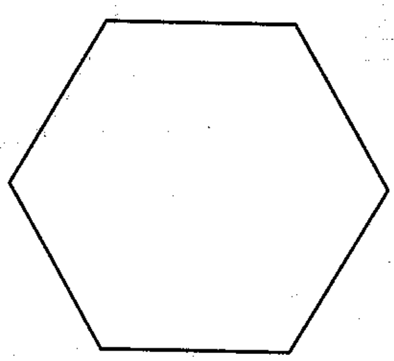

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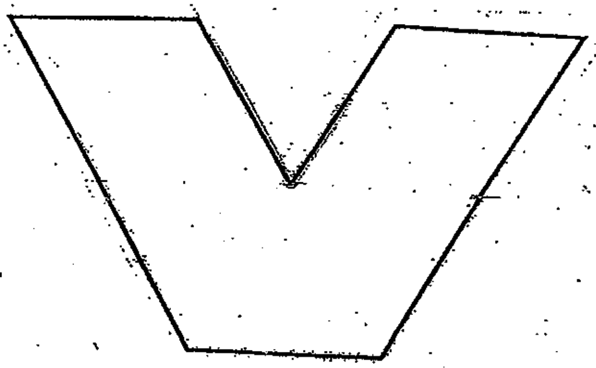
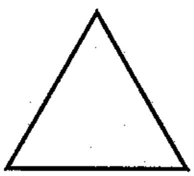
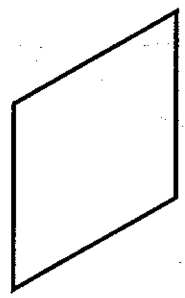
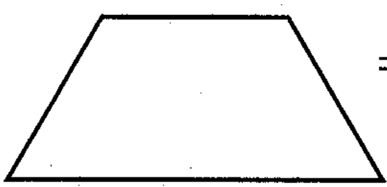
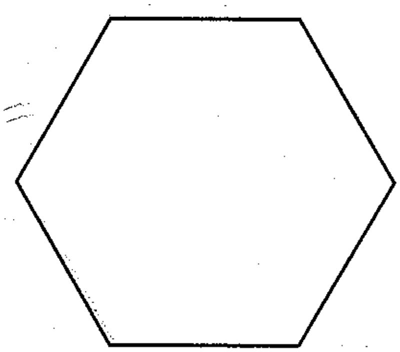
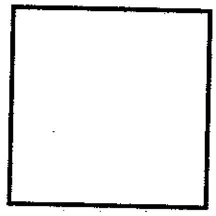
 = 1

 = ____

 = ____

 = ____

 = ____



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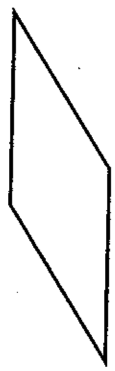

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Hint: You can use other blocks
(aside from the tan rhombus and the orange rhombus)
to figure out this relationship.

Standards for Mathematical Practice Toolkit

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COMMON CORE
STATE STANDARDS INITIATIVE
PREPARING AMERICA'S STUDENTS FOR COLLEGE & CAREER

This toolkit was created through a collaborative process involving more than 100 Preschool through 3rd grade teachers from eight Chicago Public Schools. Over a period of four days in June of 2012, these teachers met with staff and instructors of Erikson Institute's Early Mathematics Education Project in Chicago to study and discuss the eight Standards for Mathematical Practice that were published as part of the Common Core State Standards for Mathematics. We hope this toolkit can provide assistance to teachers, both in the Chicago Public Schools and elsewhere in the United States, as they shift their mathematics teaching practices to incorporate the Common Core State Standards.

Thank you to the teachers at our EME Innovations schools:

Brentano Math and Science Academy
de Diego Community Academy
Cleveland Elementary School
Gale Elementary Community Academy
Jordan Elementary Community School
Lorca Elementary School
Manierre Elementary School
Reinberg Elementary School

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COMMON CORE STATE STANDARDS FOR MATHEMATICAL PRACTICE

The 8 practice standards describe the behaviors of mathematically proficient students. Mathematics teachers at all levels should seek to develop these behaviors. Every lesson should include all the practices, though any given lesson will emphasize some practices more than others. Student behaviors are connected to teacher behaviors.

Practice #1: Make sense of problems and persevere in solving them

Students:

- Identify the important information needed to make a plan
- Monitor work throughout the process, verifying strategies and solutions
- Keep trying until a clear understanding emerges
- Show patience and a positive attitude

Teachers:

- Model how to pull out important information by asking questions and re-reading the problem carefully
- Encourage the use of different strategies and give time for students to explain strategies to one another
- Make strategic decisions about when to offer students help and when to let them struggle
- Encourage students to continue until they are confident they have done their best

Practice #2: Reason abstractly and quantitatively

Students:

- Identify relevant quantitative information in a problem situation
- Visualize a problem situation mentally
- Represent a problem and solution with pictures, models, numbers and other symbols
- Use numbers and operations flexibly

Teachers:

- Ask questions that help students abstract the math from problem situations
- Ask students to explain their thinking, regardless of accuracy
- Use thinking aloud to model reasoning
- Highlight the flexible use of numbers

Practice #3: Construct viable arguments and critique the reasoning of others

Students:

- Communicate answers and logical thinking processes using words, pictures, acting it out, etc.
- Identify confusions to discover clarity
- Ask clarifying questions to improve understanding
- Actively compare thoughts of others to own ideas

Teachers:

- Plan time for students to share and compare thinking (explain, rephrase, turn & talk, etc.)
- Establish classroom norms for the safe discussion of different ideas
- Model and encourage the asking of questions to clarify thinking
- Use confusion as an opportunity for learning

Practice #4: Model with mathematics

<u>Students</u>:

- See and describe the relationship between a model and an everyday situation
- Select and apply appropriate models to solve problems and represent thinking
- Check that models accurately reflect the situation and revise as necessary

<u>Teachers</u>:

- Plan tasks and problems that involve solving equations in everyday situations (e.g., grocery shopping, sharing)
- Provide time for students to share and discuss their models and how they relate to their thinking about a problem
- Highlight similarities and differences between various models

Practice #5: Use appropriate tools strategically

<u>Students</u>:

- Make reasonable choices about when to use tools
- Demonstrate the correct use of tools while solving problems
- Learn from the use of a tool

<u>Teachers</u>:

- Provide a variety of appropriate mathematical tools and time to explore their use
- Consistently model use of tools during instruction
- Expect students to use mathematical tools to support their reasoning

Practice #6: Attend to precision

<u>Students</u>:

- Specify the steps involved in solving a problem
- Use accurate mathematical language, including symbols, labels, definitions, and units of measure
- Calculate with precision and attention to detail

<u>Teachers</u>:

- Model clarity of explanation by using explicit language and clear mathematical models
- Encourage students to be specific when explaining their thinking
- Have students paraphrase others' thinking to push for precision and confirmation

Practice #7: Look for and make use of structure

<u>Students</u>:

- Look for and recognize mathematical significance
- Generalize relationships within and between problems (e.g., Math-to-Math connections)
- Apply a new idea to related problems

<u>Teachers</u>:

- Plan tasks and problems with patterns (e.g., number strings)
- Ask questions that focus students of the structure the problem
- Highlight different approaches for solving a problem

Practice #8: Look for and express regularity in repeated reasoning

Students:

- Check work for sense, repeatedly
- Notice patterns and connections that help them develop generalizations or "shortcuts"
- Explain what they are doing and why it makes sense
- Explain why a generalization is true and useful

Teachers:

- Ask about possible answers before, and reasonableness during and after computations
- Use thinking aloud to model how to explain what they are doing and why it makes sense
- Ask students to explain what they are doing and why it makes sense
- Ask students if a generalization is always true

Practice #1: Make sense of problems and persevere in solving them

Context: Joining problem with the change unknown

NOVICE	APPRENTICE	PROFICIENT
Student is able to identify part of the information (the given) but not enough to make sense of the problem as a change unknown problem. Student is unaware how to model or solve this type of problem. Student is not yet able to recognize whether the answer is correct or not, or gets frustrated and gives up.	Student can identify the important information in the problem but is unable to determine the correct operation for a change unknown problem. Student chooses a flawed strategy and does not verify that solution makes sense.	Student can recognize the problem type and can use the given and unknown to model the change. Student is able to check his or her work and change strategies until he or she has a clear understanding and a solution.

TEACHER QUESTIONS THAT SCAFFOLD PRACTICE #1 (in context)

- *What are the important words in the problem?*
- *How many did she have to start with and how many did she end up with?*
- *What changed to get to the final number?*
- *What ideas do you have about how to show this problem using manipulatives or pictures and how can they help us solve the problem?*
- *Would another way work better?*

Practice #2: Reason abstractly and quantitatively

Context: Mental math with number strings

NOVICE	APPRENTICE	PROFICIENT
Student may be able to solve initial problem, but cannot apply quantitative understanding to develop a strategy. Student does not find ways to use numbers flexibly to make problem-solving easier.	Student may be able to solve entire string of problems, but not be able to explain how they solved each one or why a strategy worked. Student may be able to use target strategy to solve one string, but not apply the reasoning to a similar string.	Student can successfully solve string of problems by applying their understanding of numbers and operations. Student can explain how they solved the problem and why it works.

TEACHER QUESTIONS THAT SCAFFOLD PRACTICE #2 (in context)

- *For visual number strings: Do you see any groups or patterns that could help you?*
- *For symbolic number strings: Do you see any friendly numbers that could help you, or could you make a friendly number?*
- *Can you use something you already know to help you with this problem?*

Practice #3: Construct viable arguments and critique the reasoning of others

Context: Fair share problem

NOVICE	APPRENTICE	PROFICIENT
Student cannot explain to others how he or she shared. Student cannot justify the fairness of the result. Student does not ask or respond to questions for clarity.	Student can explain some parts of sharing process, but not all. Student may have difficulty coming up with clarifying questions, even when she or he recognizes she or he does not understand. Student may be able to follow the reasoning of others but not be able to compare it to his or her own thought process.	Student communicates the strategy used to divide the total number into equal parts. Student can justify his/her reasoning and compare it to others. Student can ask and respond to questions for clarity.

TEACHER QUESTIONS THAT SCAFFOLD PRACTICE #3 (in context)

- *How do you know your answer is fair?*
- *How is what he/she did different from what you did?*
- *Which part of this doesn't make sense to you?*
- *Is there another way to show how you made it fair?*

Practice #4: Model with mathematics

Context: Classroom inventory problem

NOVICE	APPRENTICE	PROFICIENT
Student may group some objects to count but the groupings do not match his or her counting. The number does not accurately reflect the quantity of objects. Student does not recognize the mismatch between the situation and the models.	Student knows to group objects for counting but does not know why this is helpful or how groups of 10 are related to our number system. When prompted, student is able to revise model for increased accuracy. Student does not generalize models to other situations involving large counts.	Student can group objects into "friendly" number sets (e.g., 5s and 10s) to correctly count in an efficient way. Student can model counting with groups of objects and numbers and explain the relationship between the two models. Student is able to apply models to other everyday situations involving large counts.

TEACHER QUESTIONS THAT SCAFFOLD PRACTICE #4 (in context)

- *Is there an easier way to count this?*
- *Is there a way to keep track of your counting?*
- *Why did you choose to group by 10s? How is that helpful?*
- *How does your model (groupings, numbers, chart, tallies) show the total amount?*

Practice #5: Use appropriate tools strategically

Context: 1- or 2-digit addition

NOVICE	APPRENTICE	PROFICIENT
Student does not know to select a tool to help solve the problem. Student does not know which tool(s) to select in order to solve problem (e.g., blocks, number line, rekenrek). When given the appropriate tool, the student does not know how to use the tool independently.	Student realizes they have a choice of tools, but sometimes chooses a tool that does not help the student work efficiently. Student may choose an appropriate tool, but may not use it strategically (e.g., counts by ones on rekenrek). Student is sometimes able to explain why they chose a tool and how it was helpful.	Student knows how and when to use tools strategically with 1- and 2-digit addition problems. Student can accurately explain their tool choice and how it helped them solve the problem. Student's tool use enhances their understanding of the addition problem. Student is able to solve problems with multiple tools and strategies.

TEACHER QUESTIONS THAT SCAFFOLD PRACTICE #5 (in context)

- *What tool could you use to solve this problem?*
- *Why did you choose this tool?*
- *Could you show me how you used this tool?*
- *Is there another way you could use this tool?*
- *Is there another tool that may be more efficient?*

Practice #6: Attend to precision

Context: Comparison problem

NOVICE	APPRENTICE	PROFICIENT
Student cannot identify specific steps to solve the problem. Student misses key mathematical language to understand the problem. Student miscalculates or details relating to symbols, labels, or units are missing.	Student solves problem accurately, but cannot provide a detailed explanation or produces inaccurate models. Student understands the concept but miscalculates during operation or details relating to symbols, labels, or units are incorrect.	Student calculates and solves problem accurately and all details relating to symbols, labels, and units are correct. Student can precisely explain his/her thinking.

TEACHER QUESTIONS THAT SCAFFOLD PRACTICE #6 (in context)

- *What do we know? What are we trying to find out?*
- *What is the story of the problem or what is happening in this story?*
- *Have you included all the important information to communicate your answer?*
- *Can you tell us exactly what you did?*
- *How could you explain this to someone who is confused?*

Practice #7: Look for and make use of structure

Context: Function Machines (from *Everyday Mathematics* in grades K—3)

NOVICE	APPRENTICE	PROFICIENT
Student does not know to look at more than one in/out pair to find the pattern. Student does not recognize the rule and does not correctly complete the function machine.	Student can figure out the change in at least some of the in/out pairs. Student may be able to complete the function machine but has difficulty generalizing the rule to produce new in/out pairs.	Student knows to look at relationship of the in/out pairs to generalize the rule. Student can produce new in/out pairs or create his or her own function machine to fit a given rule.

TEACHER QUESTIONS THAT SCAFFOLD PRACTICE #7 (in context)

- *Do you notice a change in each pair of numbers that is the same?*
- *Did you look at all the pairs?*
- *If the pairs all change by 3, what would be another example that would work?*

Practice #8: Look for and express regularity in repeated reasoning

Context: Frames and Arrows (from *Everyday Mathematics* in grades 1—3)

NOVICE	APPRENTICE	PROFICIENT
Student cannot explain why or how the rule works. Student inserts numbers into the sequence haphazardly and does not check his or her work for reasonableness.	Student can figure out missing numbers in sequence, but cannot identify a missing rule. Solving sequences with smaller numbers is easier. Even when successful with one problem, student gets confused with new Frames and Arrows problems.	Student can complete or extend a sequence, filling in any missing parts. Student is able to explain what he or she is doing and why it works. Student can create own Frames and Arrows problems.

TEACHER QUESTIONS THAT SCAFFOLD PRACTICE #8 (in context)

- *If I put another number in the sequence, would that help?*
- *Do you see any pattern in the way the numbers change from one frame to the next?*
- *What helped you figure out the rule?*
- *How can you show that the rule works?*
- *How can you use what you learned in problem X to help you in problem Y?*

Everyday Mathematics® Goals for Mathematical Practice

The *Everyday Mathematics* authors have distilled the CCSS-M Standards for Mathematical Practice into a set of 23 "Goals for Mathematical Practice" that are intended to be more usable for elementary school teachers and students. These *Everyday Mathematics* Goals for Mathematical Practice (GMPs) provide a framework for instruction in mathematical practices similar to the framework for mathematical skills and understandings provided by the *Everyday Mathematics* Program Goals and Grade-Level Goals. The chart below provides the full text of each CCSS-M Standard for Mathematical Practice in the left-hand column along with the corresponding GMPs in the right-hand column.

Common Core State Standards for Mathematical Practice	*Everyday Mathematics* Goals for Mathematical Practice
Standard for Mathematical Practice 1: Make sense of problems and persevere in solving them.	
Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.	**GMP 1.1** Work to make sense of your problem. **GMP 1.2** Make a plan for solving your problem. **GMP 1.3** Try different approaches when your problem is hard. **GMP 1.4** Solve your problem in more than one way. **GMP 1.5** Check whether your solution makes sense. **GMP 1.6** Connect mathematical ideas and representations to one another.
Standard for Mathematical Practice 2: Reason abstractly and quantitatively.	
Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to *decontextualize*—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to *contextualize*, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.	**GMP 2.1** Represent problems and situations mathematically with numbers, words, pictures, symbols, gestures, tables, graphs, and concrete objects. **GMP 2.2** Explain the meanings of the numbers, words, pictures, symbols, gestures, tables, graphs, and concrete objects you and others use.

Common Core State Standards for Mathematical Practice	*Everyday Mathematics* Goals for Mathematical Practice
Standard for Mathematical Practice 3: Construct viable arguments and critique the reasoning of others.	
Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.	**GMP 3.1** Explain both what to do and why it works. **GMP 3.2** Work to make sense of others' mathematical thinking.
Standard for Mathematical Practice 4: Model with mathematics.	
Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.	**GMP 4.1** Apply mathematical ideas to real-world situations. **GMP 4.2** Use mathematical models such as graphs, drawings, tables, symbols, numbers, and diagrams to solve problems.
Standard for Mathematical Practice 5: Use appropriate tools strategically.	
Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.	**GMP 5.1** Choose appropriate tools for your problem. **GMP 5.2** Use mathematical tools correctly and efficiently. **GMP 5.3** Estimate and use what you know to check the answers you find using tools.

Common Core State Standards for Mathematical Practice	*Everyday Mathematics* Goals for Mathematical Practice
Standard for Mathematical Practice 6: Attend to precision.	
Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. In the elementary grades, students give carefully formulated explanations to each other. By the time they reach high school they have learned to examine claims and make explicit use of definitions.	**GMP 6.1** Communicate your mathematical thinking clearly and precisely. **GMP 6.2** Use the level of precision you need for your problem. **GMP 6.3** Be accurate when you count, measure, and calculate.
Standard for Mathematical Practice 7: Look for and make use of structure.	
Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7×8 equals the well remembered $7 \times 5 + 7 \times 3$, in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$, older students can see the 14 as 2×7 and the 9 as $2 + 7$. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(x - y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y.	**GMP 7.1** Find, extend, analyze, and create patterns. **GMP 7.2** Use patterns and structures to solve problems.
Standard for Mathematical Practice 8: Look for and express regularity in repeated reasoning.	
Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, middle school students might abstract the equation $(y - 2)/(x - 1) = 3$. Noticing the regularity in the way terms cancel when expanding $(x - 1)(x + 1)$, $(x - 1)(x^2 + x + 1)$, and $(x - 1)(x^3 + x^2 + x + 1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.	**GMP 8.1** Use patterns and structures to create and explain rules and shortcuts. **GMP 8.2** Use properties, rules, and shortcuts to solve problems. **GMP 8.3** Reflect on your thinking before, during, and after you solve a problem.

Playing Around With What We've Been Learning

Shoe Game Vignette

From Fosnot & Dolk, *Young Mathematicians at Work: Constructing Number Sense, Addition and Subtraction* (Heinemann, 2001).

"How many shoes for the people in your family, Leroy?" his teacher, Madeline asks. *"There's seven people in my family, so there are seven pairs of shoes,"* he announces with assurance, *"and that is seven left shoes and seven right shoes, and I know that seven plus seven equals fourteen."*

While playing this game, several children who have not yet constructed the logic that Leroy verbalized earlier became very excited over their discovery that the number on the die matches the number of *pairs* of shoes while the doubled total matches the number of shoes! Two children in particular, Micky and Ken, find this phenomenon magical and remain puzzled by it for a long time. Finally, Micky, with much excitement, figures it out.

"Oh wow . . . I know, Ken. Look," Micky exclaims. *"I know why this is happening. There's right shoes and left shoes! When we roll five we get to move ten. But that's five right shoes and five left shoes! But see . . . that's five pairs! It's one of each!*

Ken smiles broadly, glad that they have "cracked" the problem. *"Oh yeah, like if we get a four, we move eight shoes . . . but that's four pairs. Four left shoes and four right shoes!"*

The Shoe Game

The Shoe Game supports the development of fluency with doubles and provides children with the opportunity to notice and ponder the fact that if they roll a number and double it, the result is equivalent to moving that many pairs. For example, two fives is equivalent to five pairs of two.

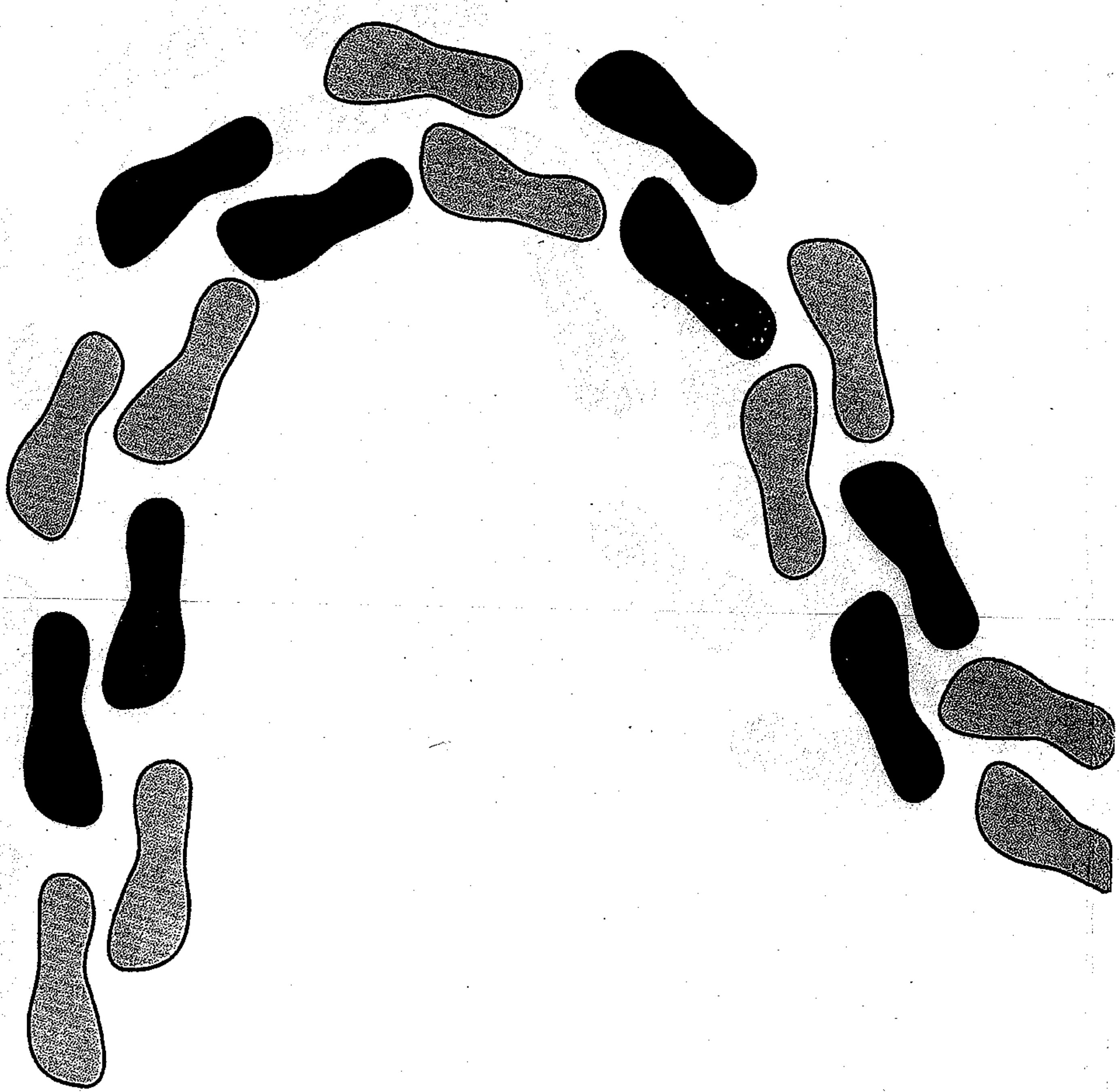
Directions for Playing

Two children play together. In turn they roll a number cube, double the number on the top of the cube, and move their game pieces that many spaces (single shoes) on the board (Appendix K). If doubling the roll is difficult for children, the second number cube (preferably of a different color) can be used to match the first and the total can then be counted. At the end of the track, the playing piece is turned around and goes back along the track to the starting point. Play continues until both players have moved their pieces back to the starting point.

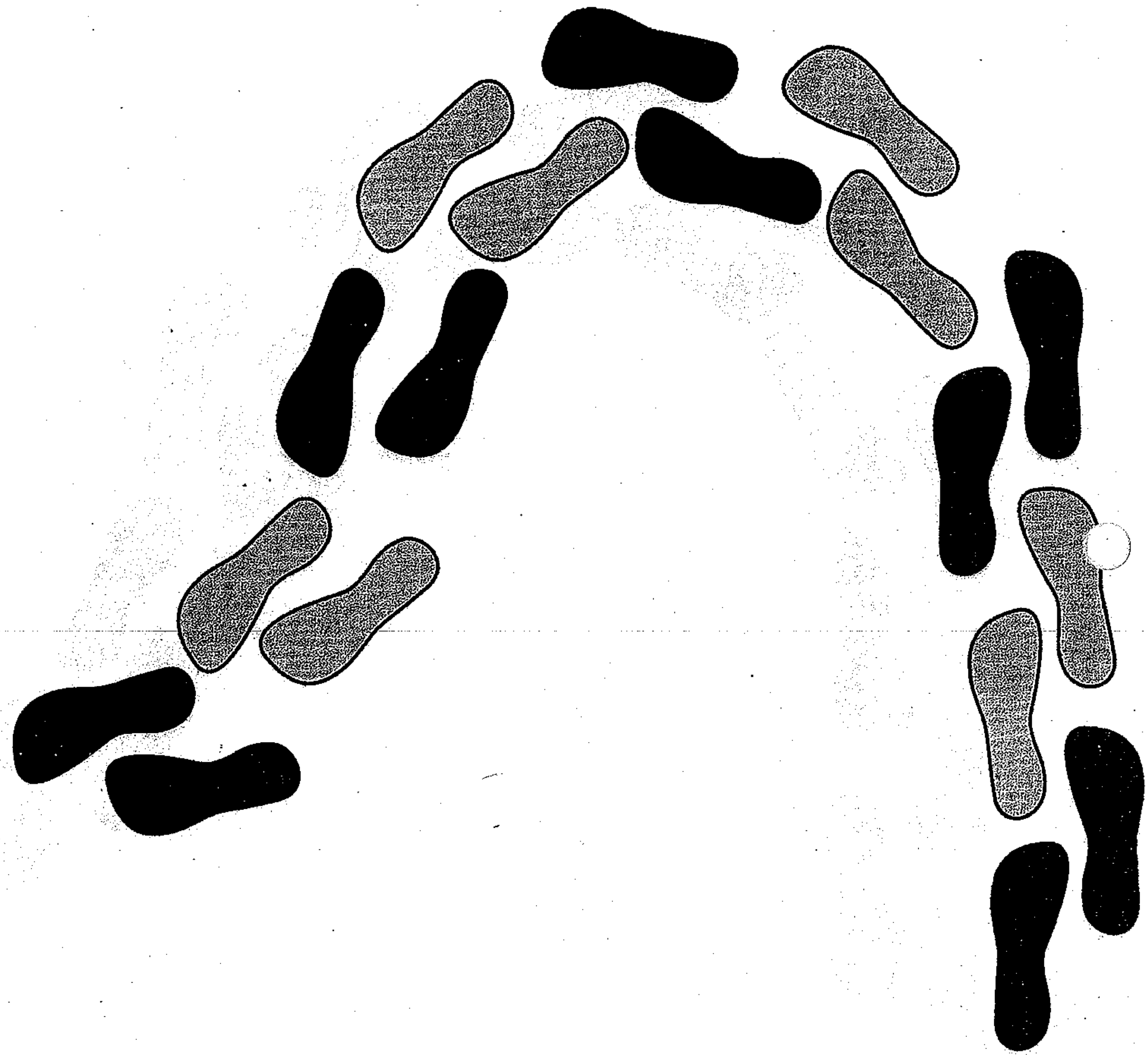
Noting the Mathematical Landscape: Openings and Possibilities

The game board has pairs of shoes (like footprints) along a track. Because players are moving by doubles, the shoe they land on should be the second of a pair. They get a color clue if they miscount and thus land otherwise, and this can become the topic of a conversation about pairs or about doubles. Note whether children can double amounts. Do they count on, know the double by heart, or need to count by ones from the beginning? As they move along the track, do they skip shoes, double count, or use one-to-one tagging carefully? For children who have some ease with doubling, engage them in wondering why it is that when they double a number, they actually move that many pairs of shoes as well—for example, 2 fives is equivalent to five pairs of shoes.

The Shoe Game game board



The Shoe Game game board



Rekenrek Bingo

The Dutch word for the arithmetic rack is *rekenrek* (pronounced *ray-ken-reck*). Directly translated, it means calculating rack. Although we have used English terminology throughout the writing of the *Contexts for Learning Mathematics* series, we have chosen to maintain the Dutch in the title of this game, as we have called it by that name since its inception. The rack employs the five- and ten-structures and supports children to use these structures to envision part-whole relations. In this Bingo game, an image is shown on the rack, supporting children in determining the sum by using the five- and ten-structures—for example, to envision 6 as 5 + 1, 8 as 5 + 3 or 10 − 2, and 8 + 7 as 10 + 5.

Materials Needed

Class-size arithmetic rack (or 20 connecting cubes strung in two rows of ten, alternating two colors in groups of five, on a board)

Arithmetic rack—one per child

Instructions for making arithmetic racks can be found in Appendix H.

Rekenrek Bingo game boards (Appendix T)—one per child

Cubes or counters

Number cards (Appendix G)—one set per group of six children

Directions for Playing

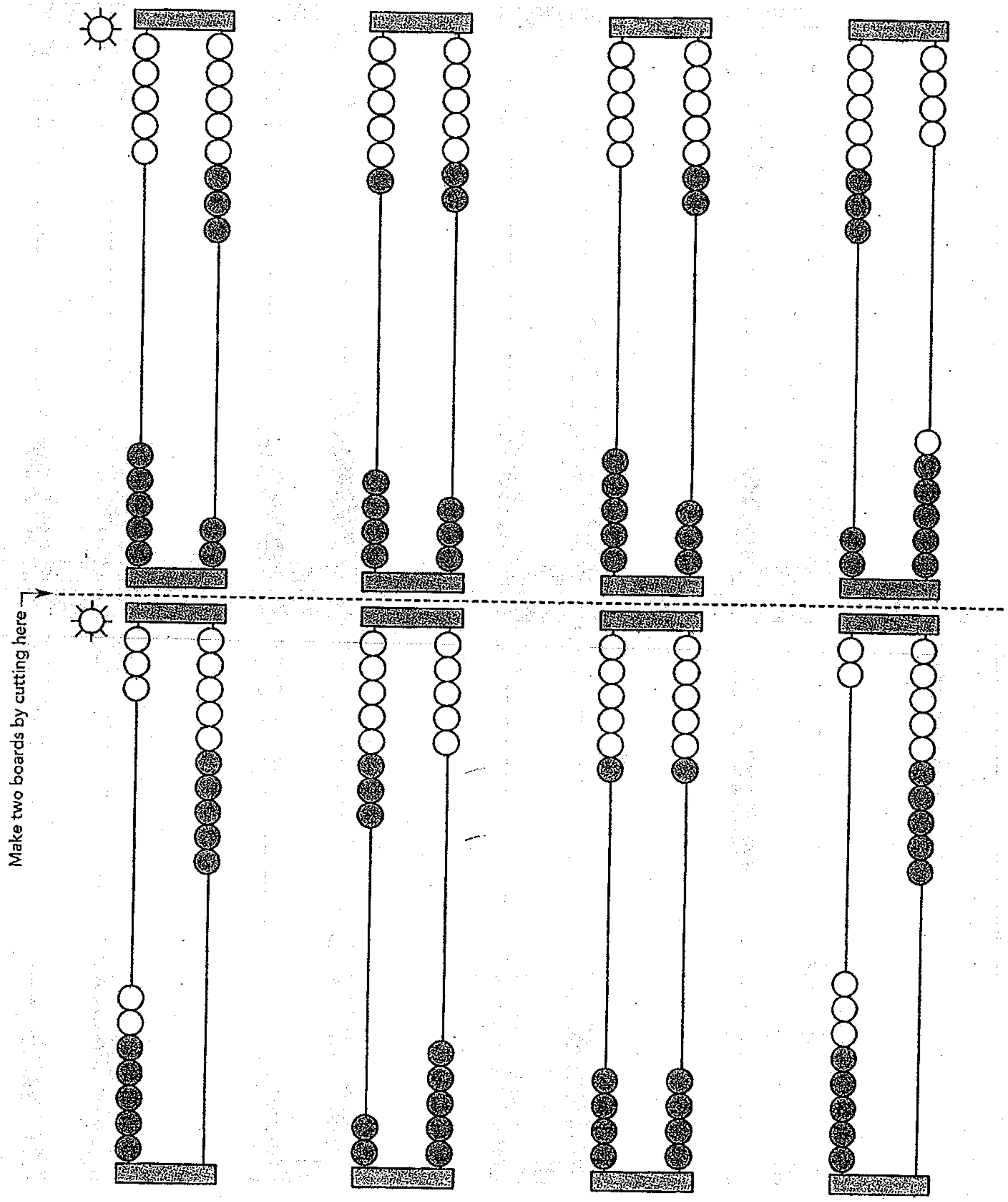
Children play the game in pairs in groups of six. Each pair has two game boards (Appendix T). Two cards are taken off the top of the deck of number cards (Appendix G) and the sum is displayed on the arithmetic rack, the arrangement of which is chosen by the teacher (or the player in charge of setting up the rack). Players then place a cube or counter on the image of an equivalent amount on the game boards. Children are not limited to covering just one image. For example, if the arithmetic rack image shows 7 on the top row only and none on the bottom row, players can cover an exact image of this or any other representation that equals 7, such as 5 on the top and 2 on the bottom, or 1 on the top and 6 on the bottom. Play continues in this fashion until all the game boards are filled.

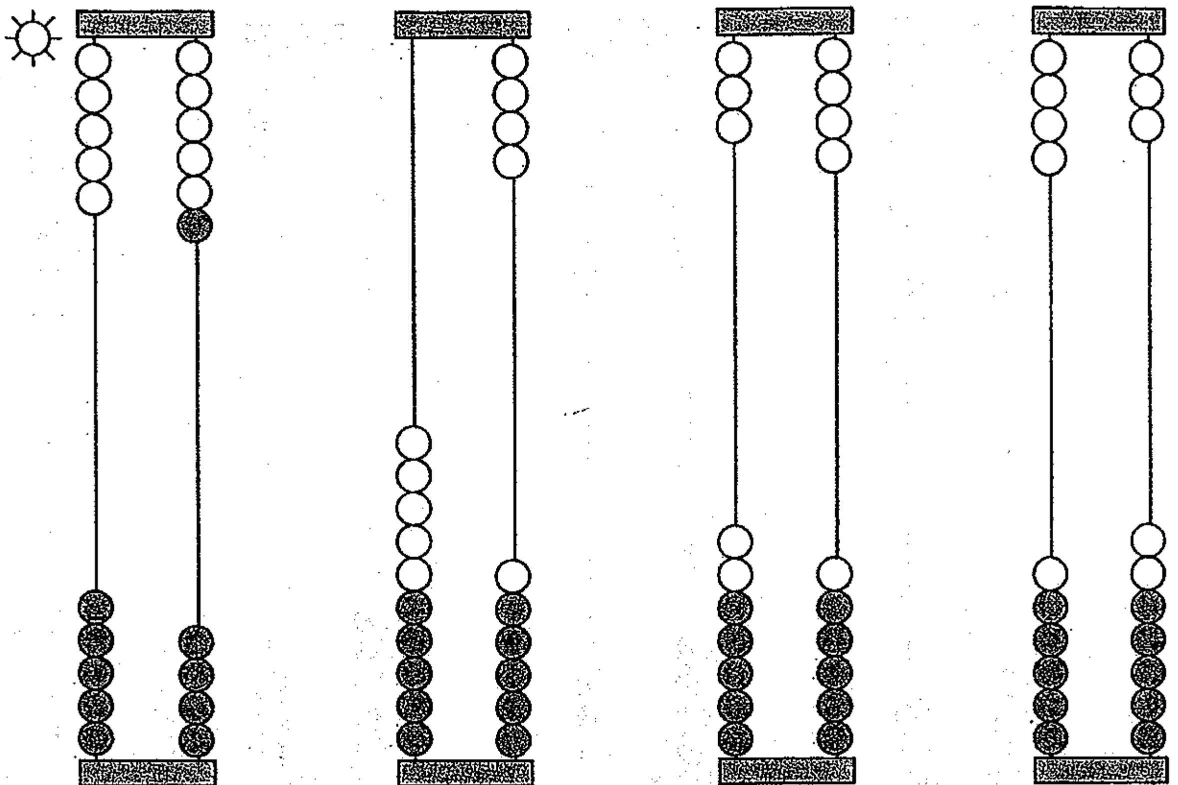
Noting the Mathematical Landscape: Openings and Possibilities

This game provides ample opportunities for children to examine equivalence and compensation as they picture transforming the image into the images on their boards. Take note of how children figure out the total number of beads both in the image shown and on their Rekenrek Bingo game boards. Do they count each bead? Do they start with the larger amount and count on? Subitize? Use the five-structure? Use landmark strategies like doubles or doubles plus or minus? Are they puzzled by equivalence, saying things like, "This looks like it has more . . . why is it the same?" Listen for children's puzzlement about each other's strategies ("He knew that was ten without counting!") and equivalence ("Why is she covering all those *rekenreks* on her board, when they don't look like what was shown?"). These kinds of comments can be the basis for a rich math congress on strategies and big ideas being developed within the classroom community.

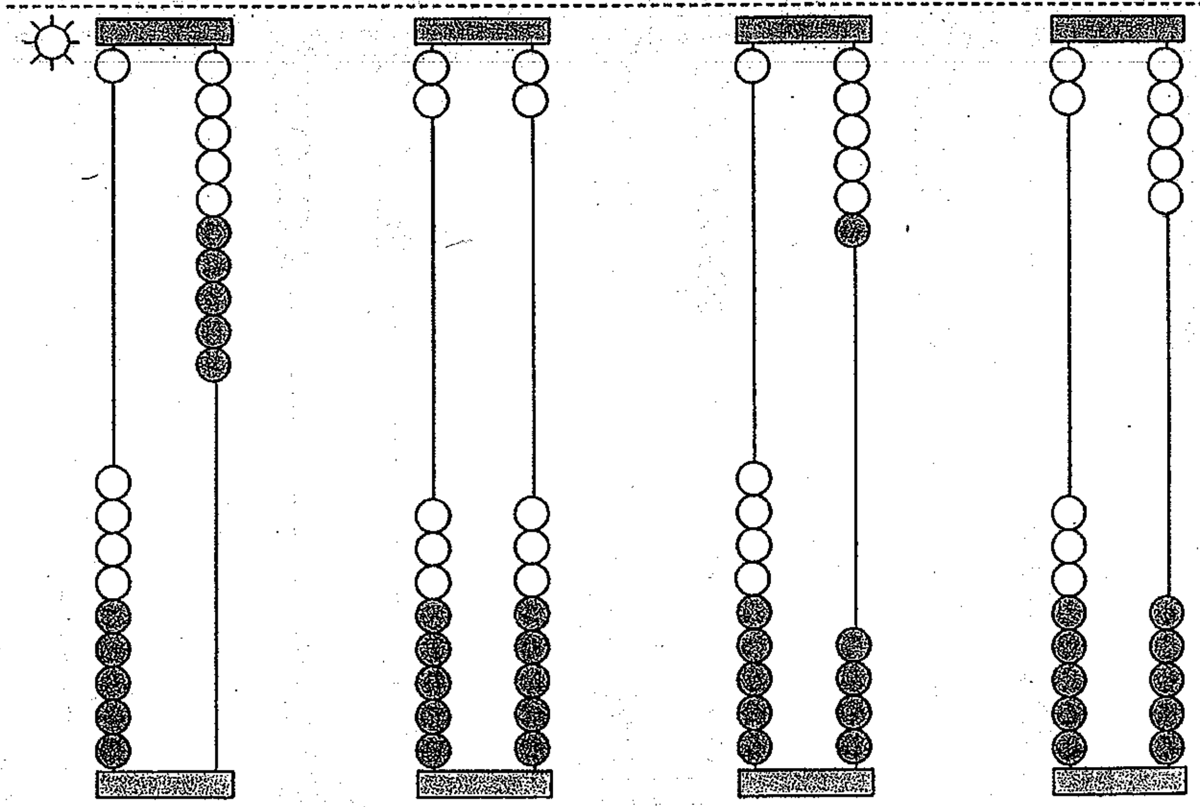
Appendix T

Rekenrek Bingo game boards



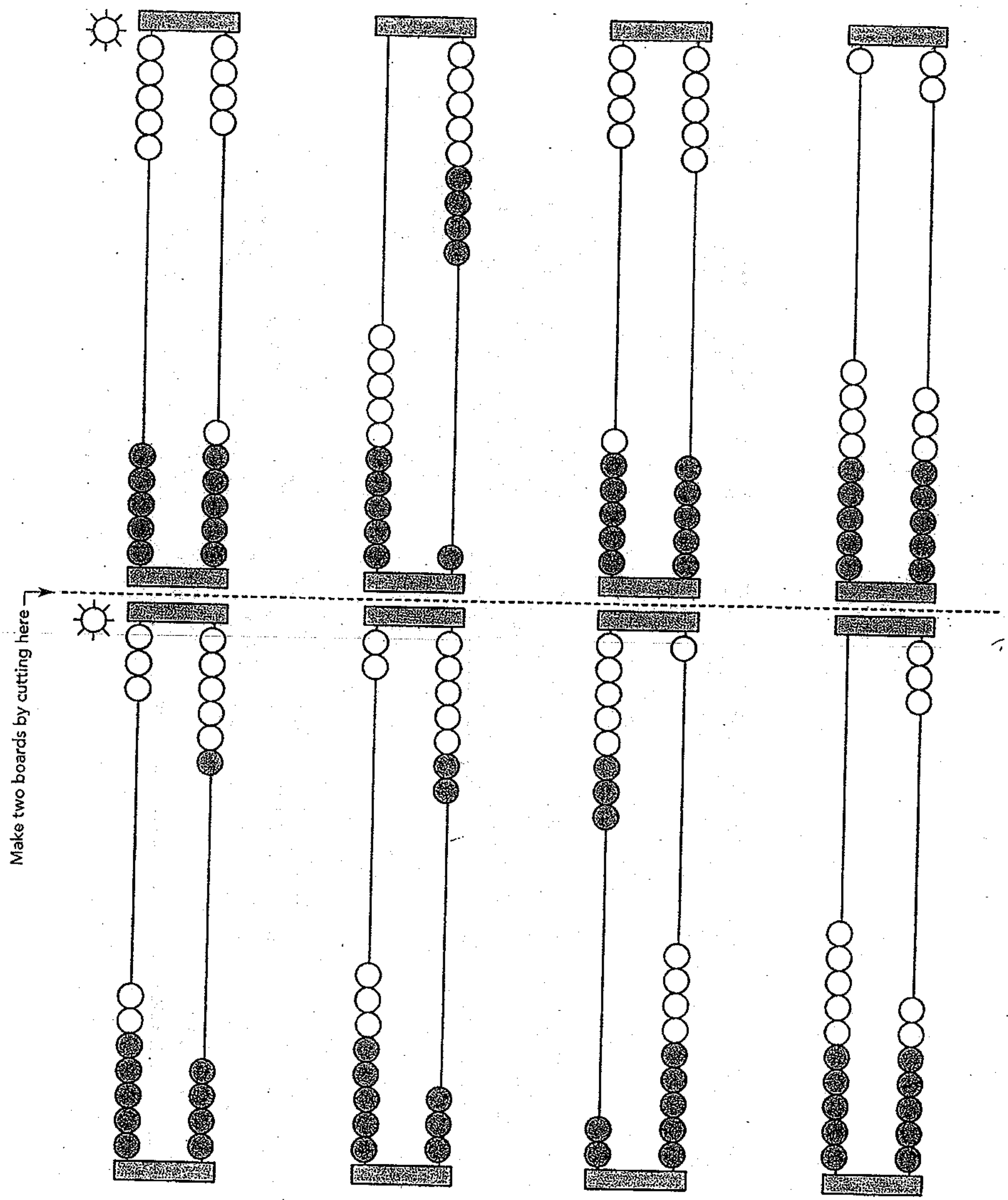


Make two boards by cutting here



Appendix T

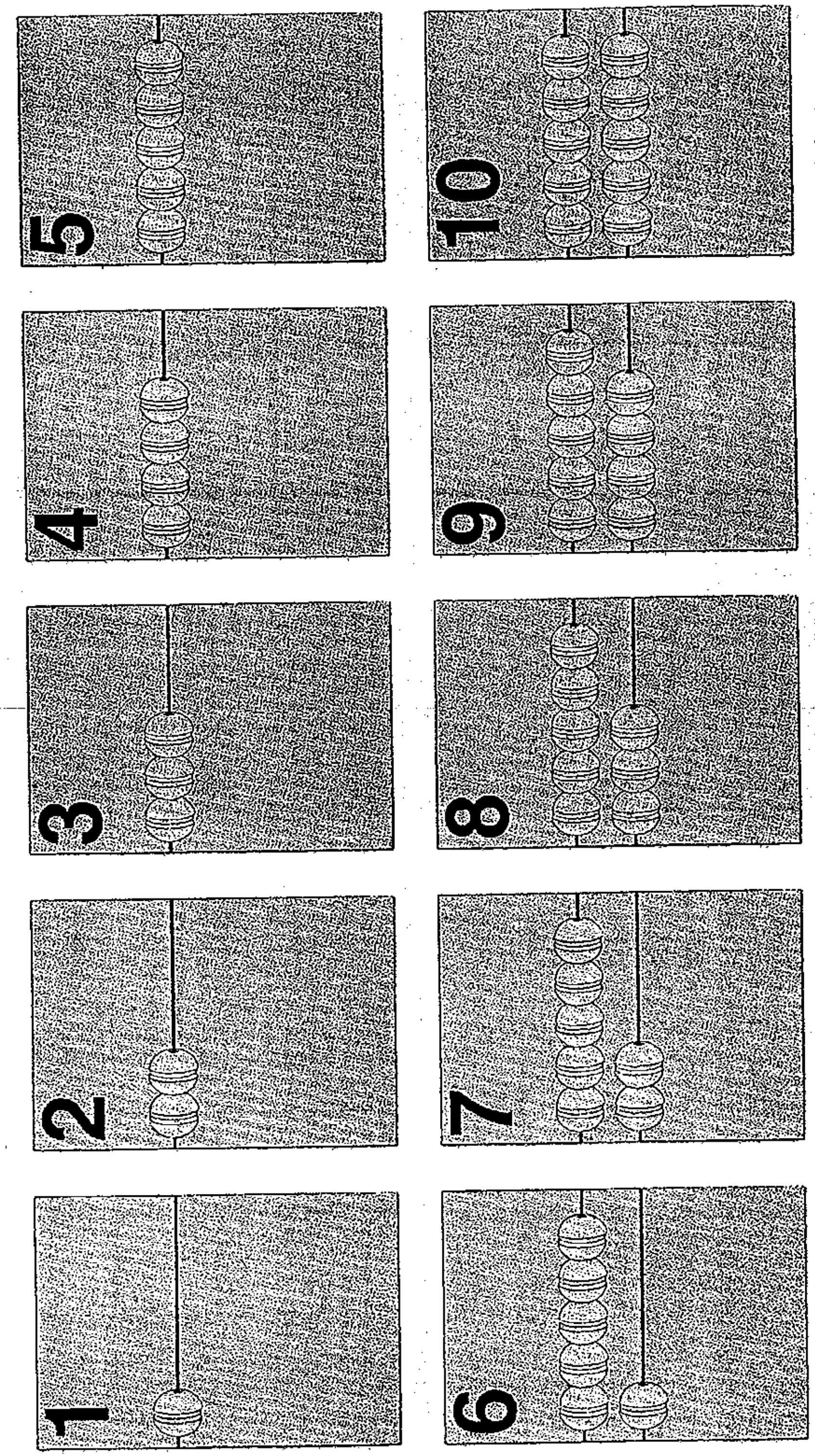
Rekenrek Bingo game boards



Appendix G

Number cards (Set I)

Make four copies. Cut out each card to make a set of 40 cards.



Passenger Pairs

The purpose of Passenger Pairs is to encourage children to examine different ways the same number of passengers can be seated on a double-decker bus, and to explore how some groupings are easier than others to recognize on the arithmetic rack because of the five- and ten-structures.

Directions for Playing

The deck (Appendix M) includes 24 cards with different groupings of bus passengers. The children play in pairs. They set up the game by mixing the cards and placing all 24 of them, face up, in four rows of six. They take turns finding two cards that depict the same number. Player One picks up two matching cards, states the number they show and how he knows they are equivalent—moving beads on the arithmetic rack to justify equivalence—and removes the cards from the array. Then Player Two takes a turn selecting two matching cards. In all cases, the partners must agree whether the cards match. The matching pairs are placed in a discard pile or turned facedown, encouraging collaborative rather than competitive play.

Noting the Mathematical Landscape: Openings and Possibilities

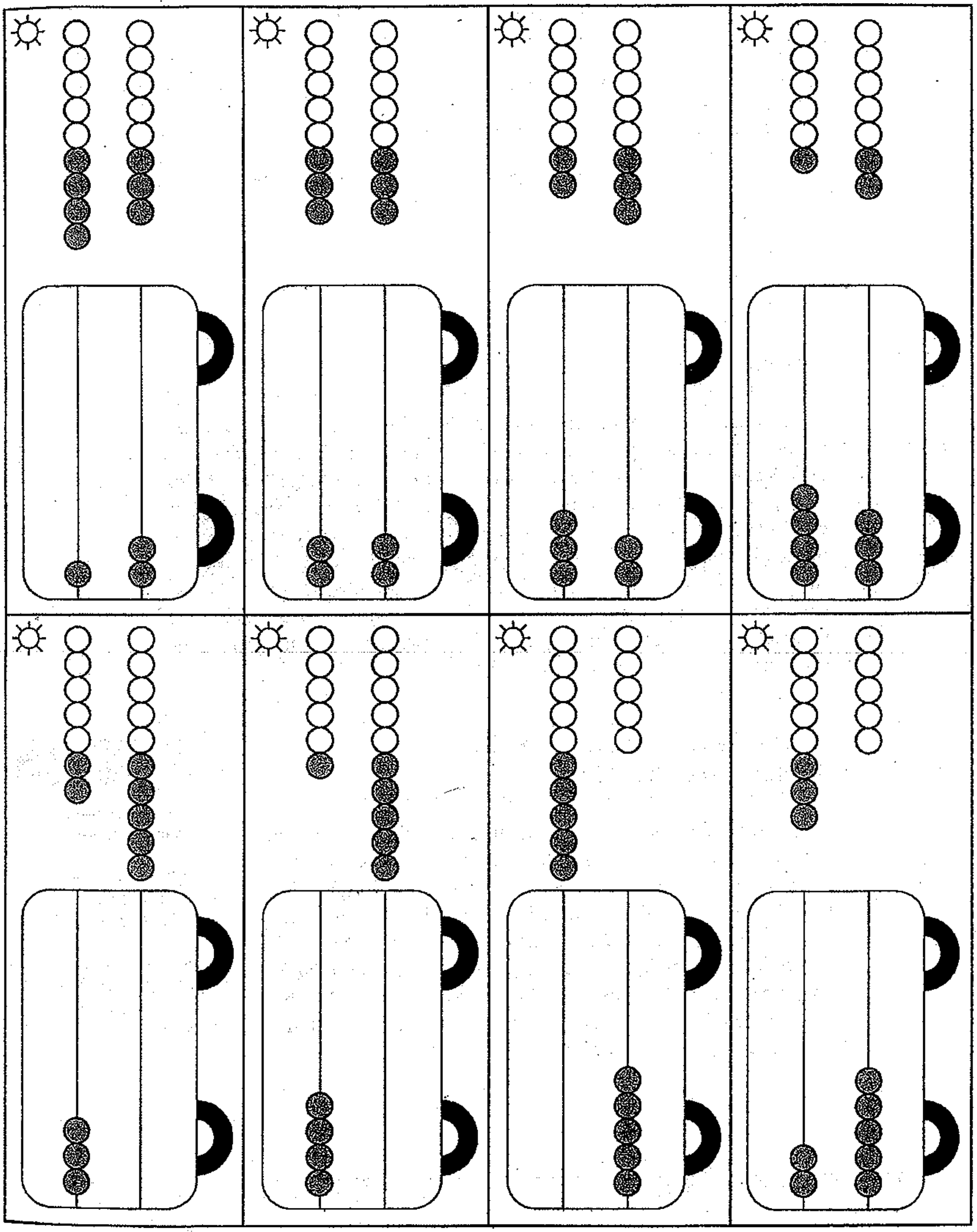
Listen in on the players' conversations. Ask children how they know how many bus passengers there are. Some children may find the number quickly but need help articulating their strategies. Note how they use the arithmetic rack. Do they easily slide beads and make use of the five- and ten-structures of the apparatus, or do they count the beads by ones? Do they use compensation easily, for example matching 5 on the top and 2 on the bottom with 4 on the top and 3 on the bottom? Do they make use of the commutative property as they search for pairs?

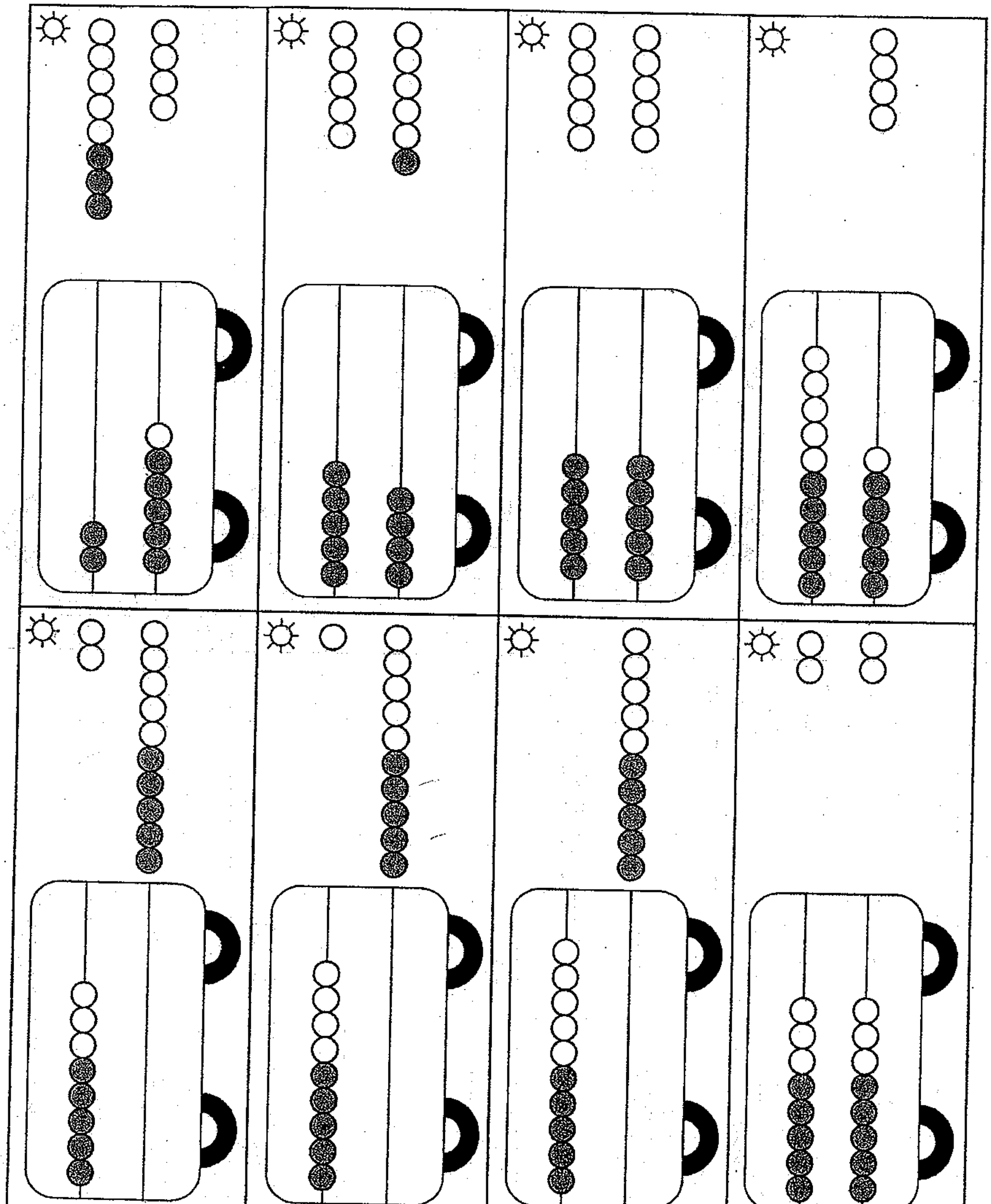
Materials Needed

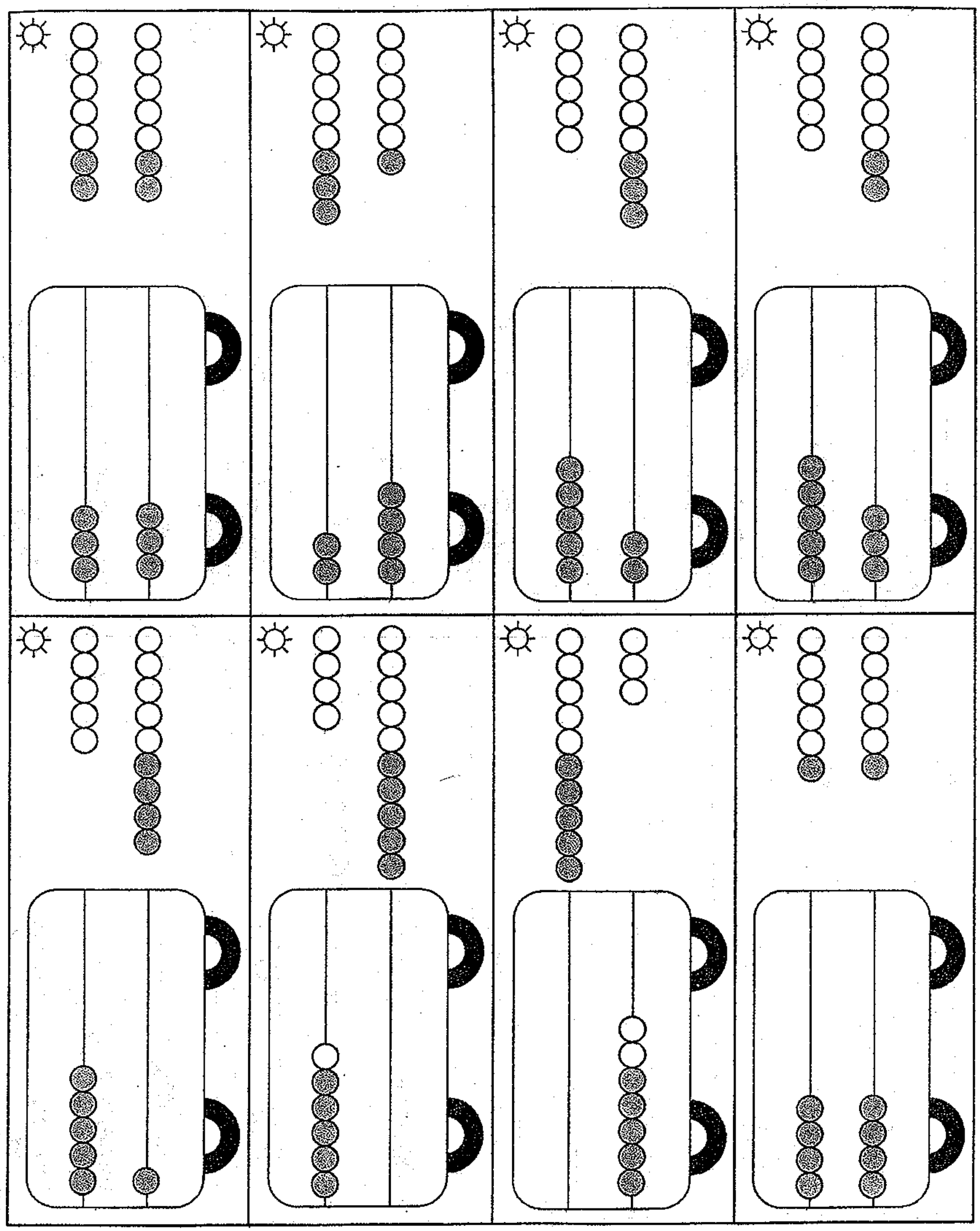
Arithmetic rack—one per pair of children

Instructions for making arithmetic racks can be found in Appendix H.

Passenger Pairs game cards (Appendix M)—one set per pair of children







Dot Card Visual Number Sequences

From *Number Sense Routines*, J. Shumway (Stenhouse, 2009)

Directions

The dot cards are to be used as Quick Look images. Show each card for a few (2-3) seconds to see if the children can determine how many.

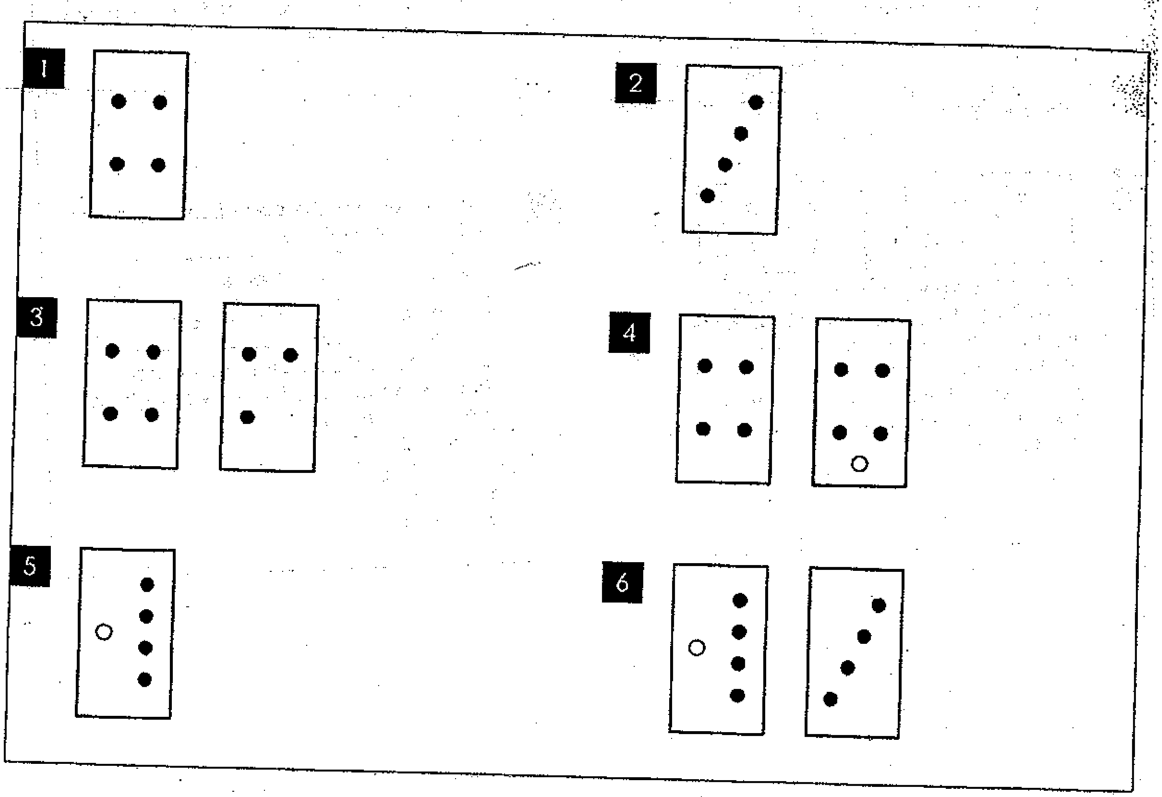
1. Tell the students that you are going to show them a card with dots on it, but they would only get a few moments to look at it. "*When you think you know how many dots are on the card, give me a silent thumbs up*
2. Show (2 seconds) the first card in the visual number sequence.
3. After everyone has figured out how many (by a show of the silent thumbs up) ask several students what did they see.
4. As each student answers, probe their thinking to see if they saw the quantity in total or as the sum of a smaller quantity. Ask questions like

   How do you know?

   How did you see it?

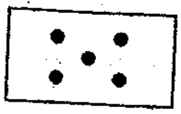
   Did you see it all together or did you see it differently?
5. Repeat with each dot card in the number sequence.

PLAYING AROUND WITH THREE, FOUR, AND FIVE



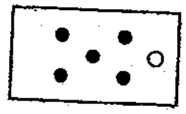
DOT CARD SEQUENCES TO TRY

1



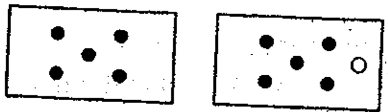
With initial dot card work, many students will be able to perceptually subitize and say "five" without counting one by one.

2



The unshaded dot highlights the idea of five and one more.

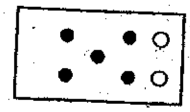
3



The combination highlights the idea that 5 + 6 = 11, because 5 + 5 = 10 and there's one extra on the second card.

5 + 5 = 10
+1 ↓ ↓ +1
5 + 6 = 11

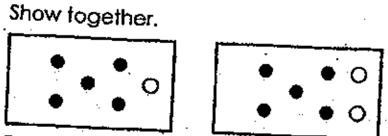
4



The two unshaded dots highlight the idea of five and two more. Students conceptually subitize seven or count up from five (⑤, 6, 7).

5

Show together.



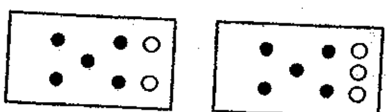
Two common ways to see this:

6 + 7 = 13 because 6 + 6 = 12, so one more is 13

5 + 5 = 10 and there are 3 extras, so 6 + 7 = 13

Dot card sequences 1 through 5 may be all you want to do in one routine. Depending on your assessments of students' thinking you could try sequences 6 and 7 next time.

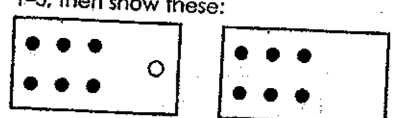
6



The 5 + 5 is highlighted to help students solve 7 + 8. If you show this combination right after the sequence 5 (the 6 + 7), some students might see that 6 + 7 can help them solve 7 + 8 and get into a discussion about the relationship between 7 + 8 and 6 + 7.

7

Review some of the cards from sequences 1–5, then show these:

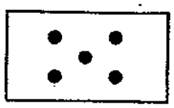


Students see the six arranged differently. The unshaded dot off to the side highlights 7 + 6 = 13, because

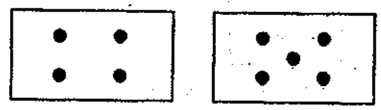
6 + 6 = 12
+1 ↓ ↓ +1
7 + 6 = 13

DOUBLES-PLUS-ONE AND MINUS-ONE SEQUENCES

1



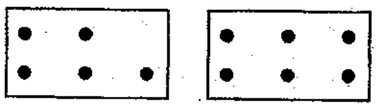
2



Two common ways to see this:

4 + 5 = 9	4 + 5 = 9
because	because
4 + 4 = 8	5 + 5 = 10
and there's one more	and there's one less

3



5 + 6 = 11

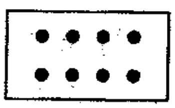
because 6 + 6 = 12

−1 ↓ ↓ −1

5 + 6 = 11

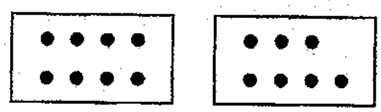
Some students might see groupings in fours and do 4 + 4 = 8.
There are still three more, so 8 + 3 = 11.
When children say this, I record it on the board like this:
4 + 4 = 8 ⟶ 8 + 3 = 11

1



Children often see 4 + 4 or count by twos.

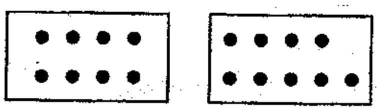
2



8 + 7 = 15

Children might use 8 + 8 = 16 and take one off or 7 + 7 = 14 and put one back on.

3



8 + 9 = 17

This combination encourages children to use 8 + 8 to help them solve 8 + 9, but another great strategy children use is to add one to the 9 to make 10.

8 + 9

+1 ↓

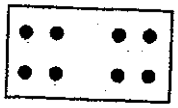
8 + 10 = 18

−1 ↓ ↓ −1

8 + 9 = 17

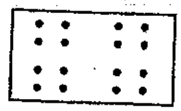
PLAYING WITH MULTIPLICATIVE IDEAS (SKIP-COUNTING, GROUPINGS, MULTIPLES)

1



Students might see two groups of 4 or four groups of 2, although this arrangement highlights two groups of 4.

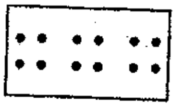
2



Many ideas and ways of thinking will come out of this card. A sample of some:

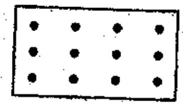
Four groups of 4	4 + 4 is 8 and 8 + 8 is 16 (double-double)	$4 \times 2 = 8$ $4 \times 2 = 8$ so $4 \times 4 = 16$

1



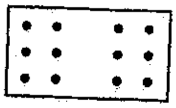
three groups of four
4, 8, 12
$3 \times 4 = 12$

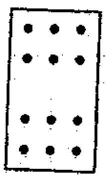
2



four groups of three
3, 6, 9, 12
$4 \times 3 = 12$

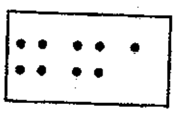
3





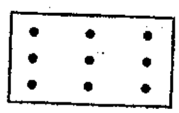
two groups of six
6, 12
$2 \times 6 = 12$

1



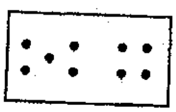
4, 8, 9
$4 \times 2 = 8$ → plus one more is 9

2



3, 6, 9
$3 \times 3 = 9$

3



$5 \times 2 = 10$ → minus one is 9
$4 \times 2 = 8$ → plus one is 9

These are all ways to "see" nine.

Making Tens: Rekenrek Number Strings

From *Number Talks* by S. Parrish (MathSolutions, 2010)

DIRECTIONS:

The following rekenrek number strings are each designed to be used in a single session, in any order. Rekenrek number strings consist of 3 – 5 problems, each labeled A, B, C, and so on. The sequence of problems within a given number string allows students to apply strategies from previous problems to subsequent problems

- Show the problems in the string one at a time on a rekenrek.
- As each problem is shown on a rekenrek, ask students, *How many beads do you see? How do you see them?*

A. 5 on top 5 on bottom **B.** 5 on top 6 on bottom **C.** 7 on top 5 on bottom	**A.** 5 on top 5 on bottom **B.** 5 on top 7 on bottom **C.** 5 on top 9 on bottom	**A.** 8 on top 2 on bottom **B.** 8 on top 3 on bottom **C.** 8 on top 4 on bottom
A. 10 on top 0 on bottom **B.** 9 on top 1 on bottom **C.** 9 on top 2 on bottom	**A.** 9 on top 1 on bottom **B.** 9 on top 3 on bottom **C.** 9 on top 5 on bottom	**A.** 7 on top 3 on bottom **B.** 7 on top 4 on bottom **C.** 7 on top 5 on bottom
A. 8 on top 2 on bottom **B.** 8 on top 4 on bottom **C.** 8 on top 6 on bottom	**A.** 7 on top 3 on bottom **B.** 7 on top 6 on bottom **C.** 7 on top 9 on bottom	**A.** 6 on top 4 on bottom **B.** 6 on top 6 on bottom **C.** 6 on top 8 on bottom

Rekenrek Number Strings: Doubles & Near-Doubles

From *Number Talks* by S. Parrish (MathSolutions, 2010)

DIRECTIONS:

The following rekenrek number strings are each designed to be used in a single session, in any order. Rekenrek number strings consist of 3 – 5 problems, each labeled A, B, C, and so on. The sequence of problems within a given number string allows students to apply strategies from previous problems to subsequent problems

- Show the problems in the string one at a time on a rekenrek.
- As each problem is shown on a rekenrek, ask students, *How many beads do you see? How do you see them?*

A. 3 on top
3 on bottom
B. 4 on top
3 on bottom
C. 3 on top
2 on bottom

A. 4 on top
4 on bottom
B. 5 on top
4 on bottom
C. 4 on top
3 on bottom

A. 5 on top
5 on bottom
B. 6 on top
5 on bottom
C. 5 on top
4 on bottom

A. 6 on top
6 on bottom
B. 7 on top
6 on bottom
C. 6 on top
5 on bottom

A. 7 on top
7 on bottom
B. 8 on top
7 on bottom
C. 7 on top
6 on bottom

A. 8 on top
8 on bottom
B. 9 on top
8 on bottom
C. 8 on top
7 on bottom

A. 9 on top
9 on bottom
B. 10 on top
9 on bottom
C. 10 on top
10 on bottom

A. 5 on top
5 on bottom
B. 7 on top
5 on bottom
C. 5 on top
5 on bottom
D. 5 on top
3 on bottom

A. 6 on top
6 on bottom
B. 8 on top
6 on bottom
C. 6 on top
6 on bottom
D. 6 on top
4 on bottom

Table Participants:

____________________ ____________________

____________________ ____________________

____________________ ____________________

Dot Cards: What mathematical strategies are these visual strings encouraging and what Big Ideas must children understand to be able to respond?

-
-
-
-

Number Strings: What mathematical strategies do these number strings encourage? What must children know to be able to use this strategy: (Write down the number string and the strategy)

Games: Noting the Mathematical Openings and Possibilities this game offers –

What should you be looking for and listening for as children are playing this game to assess their current mathematical understanding? Please use the landscape of learning to guide your responses.

- Shoe Game

- Rekenrek Bingo

- Passenger Pairs

Rekenrek: What mathematical strategies do these lessons support? What must children know to be able to use these strategies?

Making 10

Combinations! (10-20)

Combinations!
10-20

DIRECTIONS:

In this lesson, the students will work together, first with the teacher and then with a partner to create a number.

In modeling the first number, the teacher controls the top row and the student controls the bottom row.

Begin with the following example:

- *We are going to work together to build the number 15. I will start. I push 8 beads over on the top row.*
- Demonstrate. Encourage students to mirror your action, pushing over 8 beads (5red, 3 white) on the top of their Rekenrek as well.
- *Now it's your turn. How many do you need to push to the left* ***on the bottom row*** *so that we can make the number 15?*
- Students complete the problem, pushing over 7 beads on the bottom.
- Check the thinking of your students by asking questions like *How did you know that you needed to slide 7 beads more?*
- Listen for student responses. Encourage children to NOT count by ones

Now have students work in partner groups with one person controlling the top row and the second person working the bottom row. After two turns, have the partners switch roles, with the first person working the bottom row and the second person controlling the top row.

- *Let's make 13. The person on the top row needs to push some beads to the left and the person on the bottom row needs to figure out how many more beads they need to push on the bottom row to make the number.*
- After all the partner pairs have made the number, have them hold them up. Look at the different ways the students have composed 13. Ask questions like *How did you know how many beads you needed to add? How do you see the numbers?*

Repeat for the following numbers:

19
14
11
16
18
20
12
15
17

Number Strings

Number strings were supplied by - <u>Number Talks</u>, by Sherry Parrish; <u>Young Mathematicians at Work</u> by Catherine Fosnot and Maarten Dolk; Number Strings: A Work in Progress by Rebeca Itzkowich

Instructions for Number Strings

1. Write the problem on the right hand side.
2. Model thinking on the left hand side.
3. Write the problems underneath one another
4. Do the problems one at a time
5. Ask questions to verify that you are modeling their thinking correctly
6. If using a number line, if possible, do all the modeling for the string on the same number line, in different colors.
7. Verify with the student that you have represented their thinking correctly.

Example:

9+7

Student says, *"I know that 9+1 =10 so I changed the 9 to 10 and the 7 to 6"*

9 + 7
+1 -1
10 + 6 = 16

4+9

Student says, *"I know that 9+1 =10 so I changed the 9 to 10 and the 4 to 3"*

9+6

Student says, *"I know that 9+1 =10 so I changed the 9 to 10 and the 6 to5"*

8+9

Student says, *"I know that 9+1 =10 so I changed the 9 to 10 and the 8 to7"*

Jumps of 10 36+10 36+20 36+21 52+21	**Jumps of 10** 47+10 47+20 47+22 69+21
Jumps of 10 46+10 46+13 53+20 53+22 66+23	**Jumps of 10** 54+10 54+20 54+21 76+20 76+22 47+21
Splitting 218+431 123+114 337+112 232+325	**Splitting** 12+17 15+14 13+16 11+17

Splitting **18+31** **23+14** **37+12** **32+25**	**Move to friendly number** **40+4** **39+4** **39+15** **39+39**
Move to friendly number **19+1** **19+15** **19+27** **19+18**	**Move to friendly number** **79+8** **69+1+7** **69+8** **69+13**
Doubles/doubles +/- **5+5** **5+6** **6+6** **6+7** **7+8**	**Doubles/doubles +/-** **20+20** **20+22** **50+50** **50+52** **25+27**

Compensation 27+18 38+24 18+22 48+16	**Compensation** 28+25 53+18 38+47 17+78
Compensation 8+7 17+8 13+17 17+18	**Constant difference** 50-25 51-26 52-27 49-24 65-40 62-37 73-48
Constant difference 85-40 84-39 83-38 96-50 93-47 72-28	**Constant difference** 70-35 69-34 71-36 81-46 74-39 127-52

Relating addition and subtraction	**Relating addition and subtraction**
6+6 **12-6** **8+2** **10-8** **14-7**	**7+3** **10-3** **6+4** **10-4** **10-8**
Swapping	**Swapping**
34+19 **39+14** **71+26** **76+21**	**29+93** **99+23** **129+92** **199+22** **139+95**