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INNOVATIONS

Learning Lab #5

Agenda

- Using What We Have Been Learning
- Investigation: Tables & Chairs
- Operations & Number Stories: Grouping
  - Mathematical Models
- Video Analysis
  - Models of Base 10
- High-Impact Strategies for Mathematics
  - Teachers model students' thinking.
- Early Math Online
- High-Quality Books to Spark Mathematical Thinking & Action

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Euclid Catering Company has very specific rules about how to set up tables & chairs.

- All the tables are square card tables and they must be pushed together into one rectangle, so there is space in the middle for the serving platters.
- One chair must be put on each exposed side of a table.

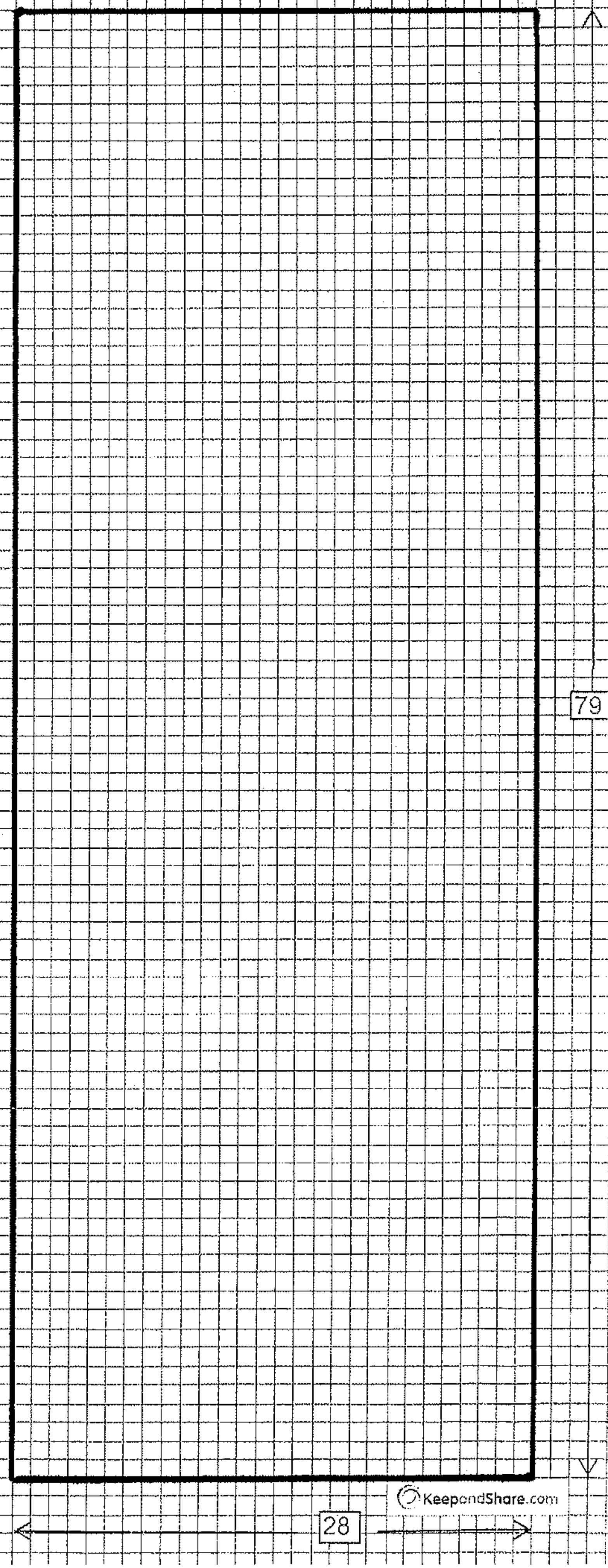
Given these rules, draw some ways to arrange tables to seat a group of 28 people. (You can use the grid paper provided on the table.)

This month is very busy for *Euclid Catering Company*! They have 79 events, each serving 28 guests. They want to be sure they order enough of their specially monogrammed napkins so that each guest at every event gets one. How many napkins will they need this month?

Try to solve this napkin problem by yourself. Show all your work.

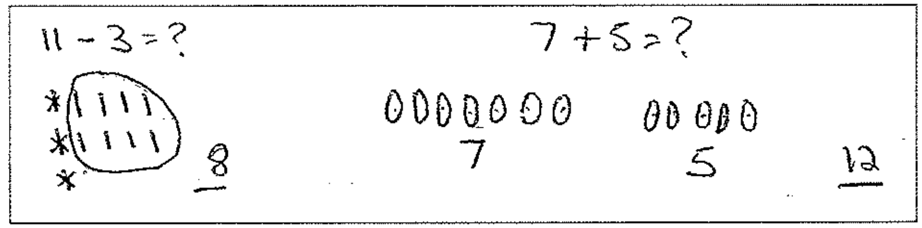
To solve the problem of 28 times 79, you could not rely on a remembered fact, but had to do the problem in several steps. The rectangle to the right is 28 squares by 79 squares and represents the expression "28x79".

Can you make smaller rectangles within the larger one to represent the different steps you took to find the answer?

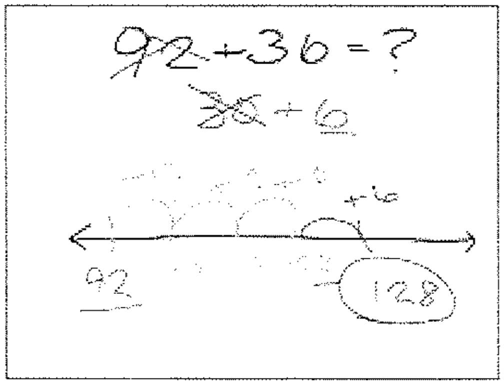


Modeling Students' Mathematical Thinking

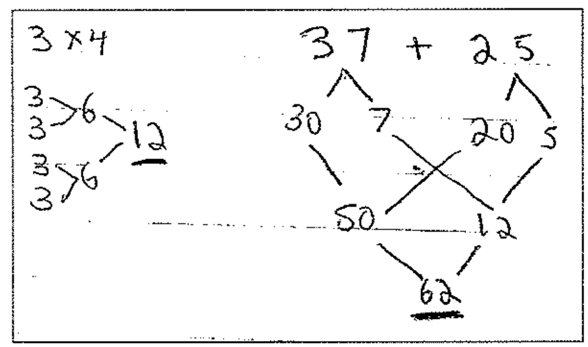
tallies or other marks



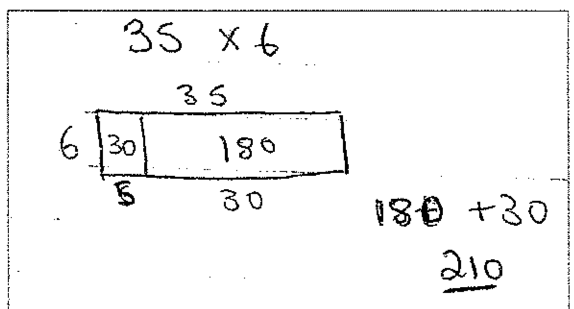
number line



number tree



rectangle or array



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Modeling Student Thinking About "*five times six*"

There are a variety of problem situations in which children might want to figure out "*five times six*" – five children each get six crackers for snack, five people riding in each of six cars, a quilt made with five rows of six squares each ...

Below are what some students said to explain how they figured out "*five times six.*" We will consider how to model the different ways the students are thinking about and solving the number sentence.

Kelly & LeShaun:

We got 5 plates, and we put 6 counters on each one, and then we counted them all. We got 30.

Devonte & Caden:

We counted by fives ... 5, 10, 15, 20, 25, 30 (Putting up a finger for each count)... six times! (Showing six fingers.)

Jody & Maria:

We knew that two 6s are 12 and two 12s are 24. That's four 6s, and one more 6 makes 30.

Peri & Tracy:

We knew that 6 x 6 was 36, so we took away 6. That gave us 30.

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High Quality Books to Spark Children's Thinking About Number & Operations

Anno, M. *Anno's Counting Book.* HarperCollins Publishing, 1975.

Baker, K. *Quack and Count.* Voyager Books, 1999.

Bang, M. *Ten, Nine, Eight.* Greenwillow Books, 1983.
También en español: Bang, M. *Diez, Nueve, Ocho.* Mulberry.

Burns, M. *Spaghetti and Meatballs for All!* Scholastic, 1997.

Carle, E. *1, 2, 3 to the Zoo.* PaperStar, 1968/1987.

Crews, D. *Ten Black Dots.* Greenwillow Books, 1986.0
También en español: Crews, D. *Los Diez Puntos Negros.* Greenwillow Books.

Cuyler, M. *100th Day Worries.* Scholastic, 2000.

Demi. *One grain of rice: A mathematical folktale.* Scholastic, 1997.

Ehlert, L. *Fish Eyes.* Sandpiper, 1990.

Fleming, D. *Count!* Henry Holt, 1992.

Giganti, P. *Each orange had 8 slices: A counting book.* Greenwillow, 1992.

Giganti, P. *How many snails? A counting book.* Greenwillow, 1988

Guy, G.F. *Fiesta!* [In English & en español] Scholastic, 1996.

Harris, T. *Splitting the Herd.* Millbrook, 2008.

Harshman, M. *Only one.* Scholastic, 1993.

Hoban, T. *26 letter and 99 cents.* Greenwillow Books, 1987.

Hoban, T. *Count and See.* Simon & Schuster, 1972

Hoban, T. *Let's Count.* Greenwillow Books, 1999.

Hoban, T. *More, Fewer, Less.* Greenwillow Books, 1998.

Hong, L.T. *Two of everything.* Albert Whitman & Company, 1993.

Hopkins, L.B. & Barbour, K. *Marvelous Math: A Book of Poems.* Aladdin, 1997.

Hutchins, P. *The Doorbell Rang.* Mulberry, 1986.
También en español: Hutchins, P. *Llaman a la Puerta.* Mulberry.

Jonas, A. *Splash!* Greenwillow Books, 1995.

Kellogg, S. *How Much is a Million?* Harper Collins Publishing, 1985.

Lewis, J.P. *Arithmetickle.* Harcourt, 2002.

Mahy, M. *17 Kings and 42 Elephants.* Dial Books, 1987.

McKissack, P.C. *A Million Fish ... More or Less.* Scholastic, 1992.

McMillan, B. *Eating fractions.* Scholastic, 1991.
También en español: McMillan, B. *A Comer Fractiones!* Scholastic.

Merriam, E.*12 Ways to Get to 11.* Aladdin Paperbacks, 1993.

Murphy, F. *Ben Franklin and the Magic Squares*. Random House, 2001.

Neuschwander, C. *Amanda Bean's Amazing Dream.* Scholastic, 1998.

Nolan, H. *How much, how many, how far, how heavy, how long, how tall is 1000?* Scholastic, 1995.

Pallotta, J. *Apple Fractions*. Scholastic, 2002.

Pallotta, J. *Count to a Million*. Scholastic, 2003.

Pallotta, J. *One Hundred Ways to Get to 100*. Scholastic, 2003.

Princzes, E. *One Hundred Hungry Ants.* Houghton Mifflin Company,1993.

Princzes, E. *A Remeinder of One.* Houghton Mifflin Company,1995.

Sayre, A.P. & Sayre, J. *One is a Snail, Ten is a Crab.* Candlewick, 2003.

Schlein, M. *More than one.* Scholastic,1996.

Tang, G. *The best of times.* Scholastic, 2002. (ages 7-12)

Tang, G. *The Grapes of Math.* Scholastic, 2001. (ages 7-12)
También en español: Tang,G. *Come Una y Cuenta 20.* Everest

Tang, G. *Math appeal: mind-stretching math riddles..* Scholastic, 2003. (ages 7-12)

Tang, G. *Math fables: lessons that count.* Scholastic, 2004. (ages 3- 6)

Tang, G. *Math for All Seasons.* Scholastic, 2002. (ages 5-8)
También en español: Tang, G. *Un, Dos, Tres, El Ano se Fue.* Everest.

Tang, G. *Math potatoes: mind-stretching brain food.* Scholastic, 2005. (ages 8-13)

Tang, G. *Math-terpieces: the art of problem-solving.* Scholastic, 2003. (ages 5-12)

Walsh, E.S. *Mouse Count.* Harcourt, 1991
También en español: Walsh, E.S. *Cuenta Ratones.* Fondo de Cultura Economica.

Smart

My dad gave me one dollar bill
'Cause I'm his smartest son,
And I swapped it for two shiny quarters
'Cause two is more than one!

And then I took the quarters
And traded them to Lou
For three dimes -- I guess he don't know
That three is more than two!

Just then, along came old blind Bates
And just 'cause he can't see
He gave me four nickels for my three dimes,
And four is more than three!

And I took the nickels to Hiram Coombs
Down at the seed-feed store,
And the fool gave me five pennies for them,
And five is more than four!

And then I went and showed my dad,
And he got red in the cheeks
And closed his eyes and shook his head--
Too proud of me to speak!

- Shel Silverstein

What's the story?
Wisdom from Frazz

Reprinted From
Kamii, C. (2004). Young Children Continue to Reinvent Arithmetic: 2nd Grade (2nd edition). Teachers College Press.

CHAPTER 2

Place Value: How Is It Learned and Unlearned?

"PLACE VALUE" REFERS to the social-conventional knowledge that in "333," for example, the first 3 means three hundred (or 3 hundreds), the second 3 means thirty (or 3 tens), and the third 3 means three (or 3 ones). Place value is now taught in first grade, and again in every subsequent grade of elementary school. Research has shown, however, that most children think that the "1" in "16" means *one* until about fourth grade. These findings were first reported by Mieko Kamii (1980, 1982) and later confirmed by other researchers such as Ross (1986). In this chapter, we will review her study and Cauley's (1988), explain the nature of representation and how place value is learned, and discuss how the algorithms of "carrying" and "borrowing" "unteach" place value.

CHILDREN'S POOR KNOWLEDGE OF PLACE VALUE

Ross's Study

Ross (1986) built on Mieko Kamii's (1980, 1982) work, as well as that of Resnick (1982, 1983) and others, in a comprehensive study of children's knowledge of place value. The subjects in her study consisted of 60 children, 15 each in grades 2 through 5. Her sampling was unusual in that she randomly selected children from 33 classrooms "from the grade level enrollment lists of five elementary schools in Butte County, California. . . . The schools were selected to represent urban and rural communities, public and private funding, and diversity with respect to the mathematics textbook series used, school size, and social class" (p. 3).

In one of her tasks, Ross presented 25 tongue depressors to each child, in individual interviews, and asked him or her to count them and to "write down the number." She then circled the 5 of 25 and asked, "Does this part have anything to do with how many sticks you have?"

After the child's response, she circled the 2 and asked the same question about its meaning.

The four levels of response she found were the following:

Level 1. The child thinks that "25" stands for the whole numerical quantity, but that the individual digits have no numerical meaning.
Level 2. The child thinks that "25" stands for the whole numerical quantity but *invents* numerical meanings for the individual digits. For example, the child thinks the "5" means groups of 5 sticks, and the "2" means groups of 2 sticks.
Level 3. The child thinks that "25" stands for the whole numerical quantity and that the individual digits have meanings related to groups of tens or ones but has only a partial or confused idea of how this all works. The sum of the parts need not equal the whole. For example, the child thinks that both individual digits mean ones, or that "5" stands for tens and "2" stands for ones.
Level 4. The child thinks that "25" stands for the whole numerical quantity, that the "5" stands for ones, that the "2" stands for tens, and that the whole must equal the sum of the parts (Ross, 1986, p. 5).

As can be seen in Table 2.1 and the following statement, Ross found essentially the same thing as did Mieko Kamii (1980, 1982): "While every child in the study was able to determine the number of sticks and write the appropriate numeral, not until grade 4 did half the children demonstrate that they knew that the 5 represented five sticks and the 2 represented 20 sticks" (p. 5).

Table 2.1: Performance on Sticks Task (by number of children)

	Level of Performance			
Grade	1	2	3	4
2	5	2	5	3
3	7	1	2	5
4	0	7	0	8
5	1	4	0	10
Total	13	14	7	26

$n = 15$ for each grade

chi-square = 30.1; $df = 9$; $p < .0004$

From *The Development of Children's Place-Value Numeration Concepts in Grades Two through Five*, by S. H. Ross, 1986, paper presented at the annual meeting of the American Educational Research Association, San Francisco, p. 6. Reprinted with permission.

Cauley's Study

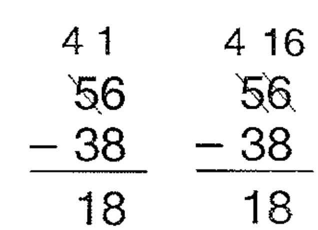
The study by Cauley (1988) is different from the preceding one in that it involved subtraction and revealed children's inability to understand place value while being able to produce correct answers. In a suburban public school and an urban Catholic school in Delaware, Cauley identified 34 of 90 second and third graders as being proficient in "borrowing." She interviewed the 34 students individually, and they produced correct answers, as shown in Figure 2.1. One of the questions she then asked was: "Before you borrowed you had [56] and after you borrowed you had this much [circling the 56 and all the borrowing marks]; did you have more before you borrowed, or after you borrowed, or was it the same?" (p. 203).

Only 41% of the 34 students replied that the number was the same after borrowing. Thirty-two percent said they had more before borrowing (because, for example, 5 is more than 4), and 24% said they had more after borrowing (because, for example, 16 is more than 6). As the 34 students comprised about a third of the 90 students, the proportion of all second and third graders who said the 56 remained the same was about 14%.

HOW IS PLACE VALUE LEARNED?

Mathematics educators generally do not differentiate between abstraction and representation and think that the use of concrete objects automatically makes an activity concrete, and that an activity is necessarily abstract if it involves written numerals. However, Piaget made a clear distinction between abstraction and representation and pointed out that children can

Figure 2.1. Children's written work in Cauley's study of subtraction with "borrowing."



use concrete objects at a high or low level of abstraction, and that they can use written symbols at a high or low level of abstraction. This theory enables us to understand that, in Ross's study, the children who said that the 2 in 25 meant "twenty" said so because they were at a higher level of abstraction than those who said that it meant "two." Let us discuss Piaget's theory of abstraction to clarify what he meant.

Piaget's Theory of Abstraction

Piaget (1978) made a distinction between two kinds of abstraction: *empirical* abstraction and *constructive* abstraction. In *empirical abstraction,* we focus on a certain property of the object and ignore other properties. For example, when we abstract the color of an object, we focus on color and ignore all other properties such as weight and the material of which the object is made (plastic, for instance).

Constructive abstraction involves the making of mental relationships between and among objects, such as "similar," "different," and "two." As stated in Chapter 1, these relationships do not have an existence in the external world. The similarity or difference between one counter and another is constructed, or mentally made, by each individual by constructive abstraction.

Constructive abstraction also is known as "reflective" or "reflecting" abstraction. The French term Piaget usually used was *abstraction réfléchissante,* which has been translated as "reflective" or "reflecting" abstraction. Piaget occasionally used the term *constructive* abstraction, which seems easier to understand. By now readers must have inferred, correctly, that logico-mathematical knowledge is constructed by constructive abstraction and empirical abstraction is involved in the construction of physical knowledge.

The conservation-of-number task demonstrates that concrete objects can be used at a high or low level of abstraction. Nonconserving children cannot conserve because their thinking is at a low level of constructive abstraction. They have the physical knowledge of the objects in the two rows but not the logico-mathematical knowledge of number. When these children reach a higher level of constructive abstraction, they begin to conserve the numerical equivalence.

Having made the theoretical distinction between empirical and constructive abstraction, Piaget went on to say that, in the psychological reality of the young child, one kind of abstraction cannot take place without the other. For example, it would be impossible for children to make the relationship "different" or "similar" (logico-mathematical knowledge) if there were no objects in their world that are different or similar (physical knowledge). Conversely, the child could not see that a counter is red (physical

knowledge) without making the category of "color" (logico-mathematical knowledge) that enables him or her to focus on color as opposed to all other properties, such as weight. Logico-mathematical knowledge (built by constructive abstraction) is thus necessary for empirical abstraction because children could not "read" facts from external reality if each fact were an isolated bit of knowledge, with no relationship to the knowledge already built and organized. This is why we said in Chapter 1 that the source of physical knowledge is only *partly* in objects and that the source of social knowledge is only *partly* in conventions made by people.

While constructive abstraction cannot take place independently of empirical abstraction up to about age 6, constructive abstraction gradually becomes independent after this age. For example, after the child has constructed number (by constructive abstraction), he or she becomes able to operate on numbers and do 5 + 5 + 5 + 5 and 4 × 5 without empirical abstraction from objects. Arithmetic and algebra are constructed by each child by making higher-level relationships out of the lower-level relationships created before.

Piaget's Theory of Representation

In empiricist thinking, it is correct to say that the symbol "+" represents addition, that the "2" in "23" represents "twenty," and that base-ten blocks represent the base-ten system. In Piaget's theory, however, all the previous statements are incorrect because representation is what a human being does. Symbols do not represent; it is always a human being who uses a symbol to represent an idea. Therefore, a human being at a low level of constructive abstraction uses symbols at a low level of abstraction. When that person reaches a higher level of constructive abstraction, he or she begins to use the same symbols at a higher level.

In Chapter 1, we saw the example of 4-year-olds who count eight objects correctly and announce that there are "eight." When asked to "show me eight," however, these children often point to the eighth object. Hierarchical inclusion of "one" in "two," "two" in "three," and so on, is achieved by constructive abstraction, and children who cannot make these mental relationships can think only about one object at a time. This is why, for them, "one" means the first object, "two" means the second object, and "eight" means the eighth object. The 4-year-olds who "count" by skipping some objects and counting others more than once likewise are using spoken numerals (social-conventional knowledge) at their low level of abstraction (logico-mathematical knowledge). When they feel the logical necessity of putting the objects into a relationship of order, they begin to count every object once and only once.

Children also represent numerical quantities by drawing pictures and writing numerals. Kato, Kamii, Ozaki, and Nagahiro (2002) showed small groups of objects (such as four dishes, six pencils, and eight small blocks) to 4- to 7-year-olds in Japan and asked them to draw/write "what's here on this sheet of paper so that your mother will be able to tell what I showed you." (In spoken Japanese, the words for "draw" and "write" sound exactly the same.) The children clearly demonstrated a close relationship between their levels of abstraction and of representation.

Those who could not make a one-to-one correspondence in the conservation task (thereby showing a low level of constructive abstraction) drew an incorrect number of objects. By contrast, most of those who could make a one-to-one correspondence in the conservation task drew the correct number of objects. Numerals were used only by conservers. An interesting finding was that 42% of those who knew how to write numerals drew pictures, revealing their preference for representing each object in the set rather than the total quantity with one numeral.

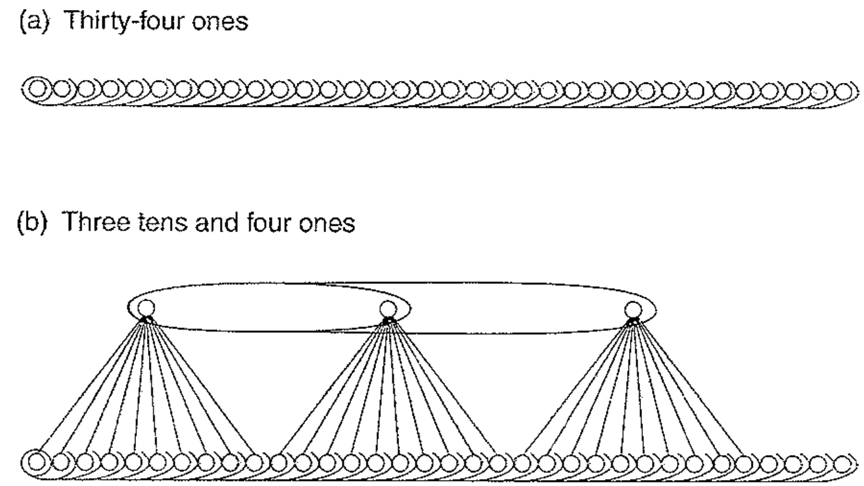
Ross (1986) gave another task using base-ten blocks to the same 60 children described earlier. She gave 40 unit blocks, several long blocks (tens), and some flat ones (hundreds) to each child and asked him or her to "use these counting blocks to build 52." Nine of the 15 second graders (60%) and most of the older students used five long blocks and two unit blocks to represent 52. If the child was successful in making 52, Ross went on to ask, "Can you find another way to represent 52?" Of the 15 students at each grade level, only 2 (13%), 8 (53%), 9 (60%), and 11 (73%), in grades 2–5, respectively, changed their initial representation to 4 ten blocks and 12 unit blocks. Only for 13% of the second graders was "ten *ones*" the same thing as "one *ten*."

Ross's two studies, Cauley's research, as well as that of Kato and colleagues demonstrate that representation is what people do at their respective levels of abstraction. Children who are at a high level of abstraction use objects (such as base-ten blocks), pictures, and numerals to represent numerical ideas at a high level of abstraction. Those who are at a low level of abstraction use the same objects, pictures, and numerals to represent numerical ideas at a low level of abstraction.

The Logico-Mathematical Knowledge of Tens and Ones

Figure 2.2(a) shows how kindergartners and most first graders think when they think "thirty-four." For them, 34 is 34 *ones*, which they constructed through constructive abstraction. Figure 2.2(b) shows how adults and older children think when they think "thirty-four." For us, 34 is 3 *tens* and 4 *ones*. What is important to note in Figure 2.2(b) is that adults and older

Figure 2.2. The structure of (a) 34 *ones* and (b) 3 *tens* and 4 *ones*.



children think "one *ten*" and "ten *ones*" *simultaneously*. Many second graders can think "one *ten*" and "ten *ones*" at two different times but not at the same time. Another task devised by Ross (1986) demonstrates the difference between young children's thinking about *tens* and *ones* at two different times and older children's simultaneous thinking.

In individual interviews, Ross presented many lima beans and 9 one-ounce plastic cups to the same 60 children described earlier and asked them to put 10 beans into each cup. When a child had four cupfuls (each containing 10 beans) and eight loose ones, Ross asked, "How many beans do you think there are here altogether?" She found the following three levels:

Level 1. The children were simply unable to count the beans.

Level 2. The children counted the beans mostly by ones, rather than using the knowledge they had that there were 10 beans in each cup.

Level 3. The children counted 40 beans by tens and then counted the rest by ones. Some of them used implicit or explicit multiplication and said, "Four tens is forty," or "Four times 10 is forty."

It can be seen in Table 2.2 that only 9 second graders (60%) counted the 48 beans correctly by tens. It also can be seen that the proportion counting by tens increased with age. Children count objects by using the num-

Table 2.2: Performance on Beans Task (by number of children)

Grade	Level of Performance 1	2	3
2	2	4	9
3	0	4	11
4	1	1	13
5	0	0	15
Total	3	9	48

$n = 15$ for each grade

chi-square = 11.0; $df = 6$; $p < .0884$

From *The Development of Children's Place-Value Numeration Concepts in Grades Two through Five*, by S. H. Ross, 1986, paper presented at the annual meeting of the American Educational Research Association, San Francisco, p. 17. Reprinted with permission.

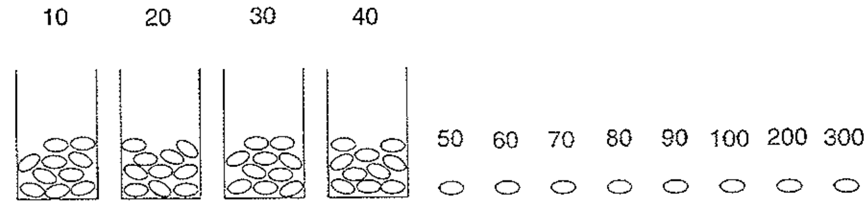
bers they have constructed in their heads, and if they have only *ones* in their heads, as shown in Figure 2.2(a), they can count only by *ones*. They can count by *tens* in this situation only if they have *tens* in their heads, as shown in Figure 2.2(b).

It is possible to teach children to count by tens. However, many children find 300 lima beans by counting as shown in Figure 2.3. These children have been taught to count by tens but cannot shift to ones after "forty" because, to shift to ones, they have to have been thinking *simultaneously* about *tens* and *ones*. This ability to think at two hierarchical levels simultaneously is logico-mathematical knowledge. How second graders in our classrooms construct this logico-mathematical knowledge is the topic to which we now turn.

The system of tens has to be constructed by each child out of his or her own system of ones. The best way to encourage children to begin to do this is to encourage them to *think*. For example, when they have to think of a "quick and easy way" to deal with 9 + 6, they will think about a *ten* and *ones* simultaneously if they know that 9 + 1 = 10. As can be seen in Chapter 6, one of the ways they invent is (9 + 1) + 5. If they are later asked for a way to deal with 19 + 6, they are likely to invent (19 + 1) + 5 or [10 + (9 + 1) + 5]. When they get to 29 + 36, they may do 20 + 30 = 50, 9 + 6 = 15, and 50 + 15 = 65; or they may change the problem to 30 + 35. When children thus struggle to *think* about *tens* and *ones simultaneously*, they are encouraged to build the mental structure illustrated in Figure 2.2(b).

Second graders already have the social knowledge that the way to write "ten" is "10," and that the way to write "twenty" is "20," and so on.

Figure 2.3. How some children count 48 beans by not shifting to *ones* after "forty."



In Chapter 6, we will show how the teacher writes and erases parts of two-digit numbers as children construct the logico-mathematical knowledge of tens and ones.

Figure 2.2(b) shows why base-ten blocks and 10 straws bundled together with a rubber band cannot empirically teach children to think "one ten" and "ten ones" *simultaneously*. By looking at a long stick with 10 segments or 10 straws bundled together, the child can think "one ten" and "ten ones" only at two different times. Adults already have the logico-mathematical knowledge of tens and ones, and this is why we can see "one ten" and "ten ones" simultaneously in a ten block.

HOW IS PLACE VALUE UNLEARNED?

When we wrote the first edition of this book in the late 1980s, we had no idea that the teaching of "carrying" and "borrowing" had the effect of "unteaching" place value. We knew that students transferring from other schools clung to the algorithms they could not explain and that their knowledge of place value was very poor. In dealing with

$$\begin{array}{r} 15 \\ \underline{+\ 27} \end{array},$$

for example, the children who had been at our school since kindergarten did 10 + 20 = 30 first because they had never been taught to do 5 + 7 first. The students who transferred to our school in second grade had already been taught to add the ones first and continued to use this method. We noticed, however, that the cognitively very advanced transfer students quickly began to add the tens first. Those in the "normal" range or below, however, continued to "carry" for a long time—until January, March, or beyond.

By May 1990, we had hypothesized that algorithms were the cause of children's poor knowledge of place value, and we interviewed all the second graders at our school, using a variety of computational problems. In May 1991, using similar problems, we interviewed the same children at the end of third grade, as well as four classes of fourth graders. An analysis of these and many other interviews clearly indicated that algorithms are harmful to children, and Chapter 3 of *Young Children Continue to Reinvent Arithmetic, 3rd Grade* (Kamii, 1994) was entitled "The Harmful Effects of Algorithms."

In retrospect, we should have thought about the harmful effects of algorithms when we wrote the first edition of this volume. However, we were blind to this possibility and believed that children who invented their own procedures were just better than the traditionally instructed children. In other words, we assumed that algorithms were good for children but that our method was better for them. We also assumed that place value was very hard for young children and never thought that instruction in a public school could be harmful to them. Many data can be presented to prove the harmful effects of algorithms, but we will limit ourselves to children's reactions to five problems.

7 + 52 + 186

There were three classes of second graders at our school in May 1990, and all the children in the three classes were individually interviewed. One of the problems given in the mental-arithmetic interview was 7 + 52 + 186 presented twice—once in vertical form and later in horizontal form. The three classes did not differ very much when the question was written vertically, but striking differences emerged when the same problem was presented horizontally.

All the answers to the horizontally written problem given by the three classes are listed in Table 2.3. The teacher of the first class (labeled "Algorithms") taught algorithms, but the teachers of the other two classes did not. The two classes differed, however, in that only the teacher (LJ) who had the class labeled "No algorithms" immediately called parents at home when children were taught to "carry" and to "borrow."

Most of the students in the "No algorithms" class typically began by saying, "One hundred eighty and fifty is two hundred and thirty" and then added the ones. This is why nearly four times as many children in the "No algorithms" class got the correct answer as those in the class labeled "Algorithms" (45% compared with 12%).

Much more informative were the *incorrect* answers the children gave. The two dotted lines across Table 2.3 were drawn to indicate a range of incorrect but reasonable answers. All the incorrect answers given by the "Algorithms" class were above and below this range and revealed poor number sense and knowledge of place value. For example, three children

Table 2.3: Answers to 7 + 52 + 186 Given by Three Classes of Second Graders in May 1990

Algorithms $n = 17$	Some algorithms taught at home $n = 19$	No algorithms $n = 20$
9308		
1000		
989		
986		
938	989	
906	938	
838	810	
295	356	617
		255
		246
245 (12%)	245 (26%)	245 (45%)
		243
		236
		235
200	213	138
198	213	—
30	199	—
29	133	—
29	125	—
—	114	
—	—	
	—	
	—	
	—	

Note: Dashes indicate that the child declined to try to work the problem.

in the "Algorithms" class got 29 or 30 for 7 + 52 + 186. These children added all the digits as *ones* (7 + 5 + 2 + 1 + 8 + 6 = 29). Many in the same class gave answers like 9308, 938, and 838 by proceeding from right to left and doing something like 2 + 6 = 8, 5 + 8 = 13, carry 1, and 1 + 7 + 1 = 9. A characteristic of this class was the children's emotional flatness, and the fact that no one seemed to feel anything wrong with answers in the 800s, 900s, and beyond. The children seemed to be functioning like machines, without any intuition or number sense.

The children in the "No algorithms" class also made errors, but their wrong answers were more reasonable because most of them got to 230 by beginning with 180 + 50 = 230, as stated earlier. The errors they made were mostly in dealing with the ones. The child who produced the answer of 617 was taught algorithms at home by parents who had said that they would stop this instruction.

The class labeled "Some algorithms taught at home" in Table 2.3 came out between the other two classes. The percentage getting the correct answer was 26, which was between 12% and 45%. The range of incorrect answers was not as outlandish as in the "Algorithms" class but not as reasonable as in the "No algorithms" class.

All the second graders were mixed before going to third grade and divided into three heterogeneous classes as randomly as possible. In May 1991, a similar problem, 6 + 53 + 185, was given to all the third- and fourth-grade classes; the results are presented in Tables 2.4 and 2.5.

Only one of the three third-grade teachers was a "No algorithms" teacher. Although she had 20 students in her class, only 13 of them had never been taught any algorithms at school, and 3 of the 13 were absent

Table 2.4: Answers to 6 + 53 + 185 Given by Three Classes of Third Graders in May 1991

Algorithms $n = 19$	Algorithms $n = 20$	No algorithms $n = 10$
	800 + 38	
838	800	
768	444	
533	344	284
246		245
244 (32%)	244 (20%)	244 (50%)
235	243	243
234	239	238
	238	
	234	
213	204	221
194	202	
194	190	
74	187	
29	144	
—	139	
—	—	
	—	

Note: Dashes indicate that the child declined to try to work the problem.

Table 2.5: Answers to 6 + 53 + 185 Given by Four Classes of Fourth Graders in May 1991

Algorithms $n = 20$	Algorithms $n = 21$	Algorithms $n = 21$	Algorithms $n = 18$
	1215		
	848		
	844		
	783		
1300	783		10,099
814	783		838
744	718	791	835
715	713	738	745
713 + 8	445	721	274
	245		
244 (30%)	244 (24%)	244 (19%)	244 (17%)
243	234		234
	224		234
			234
194	194	144	225
177	127	138	"8, 3, 8"
144	—	134	"4, 3, 2"
143	—	"8, 3, 7"	"4, 3, 2"
134		"8, 1, 7"	—
"4, 4, 4"		—	—
"1, 3, 2"		—	—
—		—	
		—	
		—	
		—	
		—	
		—	
		—	

Note: Dashes indicate that the child declined to try to work the problem.

on the day of the interview. It can be observed again that the "No algorithms" group produced the correct answer more often (50%) than the two "Algorithms" classes (32% and 20%, respectively). The incorrect answers of the "No algorithms" class were also much more reasonable than the wrong answers produced by the "Algorithms" classes.

All the fourth-grade teachers taught algorithms in 1990–91, and the children in Table 2.5 had been taught algorithms for 1 to 4 years. As far as the proportion getting the correct answer is concerned, we can see that all the fourth-grade classes did worse than the second and third graders who had never been taught any algorithms (30%, 24%, 19%, and 17%, compared with 45% and 50%). The incorrect answers produced by the fourth graders were as outlandish as those of the "Algorithms" third graders, but a new symptom appeared: answers such as "Eight, three, seven," indicating that each column remained separate in these children's minds, from right to left. These students not only had inadequate knowledge of place value but also thought about each column as an isolated column, and did not even bother to read the answer as "seven hundred thirty-eight."

By fourth grade, we expected children at least to be bothered if they got answers greater than 400 or smaller than 200. However, none of those who produced incorrect answers showed any sign of hesitation, and 19% said they could not add the three numbers without a pencil. The fourth graders who had been taught algorithms for 1 to 4 years thus can be said to have done considerably worse than the second graders who were not taught these rules.

22
+ 7 in First Grade

Several years later in early May, one of us (CK) interviewed 37 children in 2 first-grade classes, and one of the problems given was the following:

$$\begin{array}{r}22\\ \underline{+\,7}\end{array}$$

The two classes had been receiving traditional math instruction, and one of the teachers had taught the algorithm for two-digit addition without regrouping. As can be seen in Table 2.6, 67% of the class that had not been taught the algorithm produced the correct answer, but only 37% of the "Algorithm" class did.

Most of the errors made by the "No algorithm" class were 24, 25, 27, or 28. These errors were much more reasonable than answers like 11, which

Table 2.6: Answers to $\begin{array}{r}22\\ \underline{+\,7}\end{array}$ Given by Two Classes of First Graders in the Same School

Answers given	Algorithm taught $n = 19$	Algorithm not taught $n = 18$
29	7 (37%)	12 (67%)
24, 25, 27, or 28	1 (5%)	4 (22%)
11	6 (32%)	0 (0%)
10 or 12	2 (11%)	0 (0%)
"9 and 2, 92"	1 (5%)	0 (0%)
"a 2 and a 9"	1 (5%)	0 (0%)
9	1 (5%)	0 (0%)
Stuck after 2 + 2 = 4	0 (0%)	1 (6%)
Sits silently and ends up agreeing to skip problem		1 (6%)

was given by 32% of the "Algorithm" class. Since the children in the "No algorithm" class counted-on from 22, their errors were larger than 22. By contrast, 53% (32% + 11% + 5% + 5%) of the "Algorithm" class got answers smaller than 22, essentially because they treated all the digits as ones. About a third of the "Algorithm" class got the answer of 11 by doing 2 + 2 + 7 = 11. We were surprised to learn that a few short lessons on double-column addition can cause first graders to unlearn the little they knew about place value.

13
× 4 in Third Grade

In the spring of 1992, we individually interviewed 13 third graders at our school, who had never been taught algorithms at school, and 39 third graders in another school where math was traditionally taught. The interviewer wrote the multiplication problem on a blank sheet of paper and asked each child to "work this problem," offering a pen. When the child finished writing the answer, the interviewer brought out a bagful of chips and asked, "If we make four piles of 13, 13, 13, and 13 chips (indicating four different locations on the table in front of the child), will we have what this problem says?" All the children replied in the affirmative, and four piles of 13 chips were made together. The interviewer then asked, "If we pushed all these together, how many chips would we have?" All the children replied, "Fifty-two," and the interviewer asked each child to explain with the chips (which were still in four piles of 13 each) "how all this works," pointing to what the child had written.

If the child showed only 4 chips (oooo) to explain 4 × 1, the interviewer remarked, "You used all these [pointing to the chips the child had used to explain 4 × 3 and 4 × 1] to explain how 'all this' works [pointing to the child's writing]. But you didn't use any of these [pointing to the unused chips]. Were you supposed to use all of them, or were you not supposed to?" The interviewer thus asked questions that might prompt a better explanation if the child did not adequately explain the written procedure on his or her own.

As can be seen in Table 2.7, all the children in both groups wrote the correct answer of 52. Almost all the children (97%) in the comparison group used the algorithm, while none in the constructivist group did. The proportion who adequately explained all the steps of their written computation was 92% for the constructivist group and only 5% for the comparison group (a difference significant at the .001 level). The inability of the comparison group to explain the algorithm they had used correctly was due mostly to their poor knowledge of place value. Eighty-seven percent interpreted the 1 of 13 as a *one* and showed only four chips (oooo) to explain the 4 × 1 part. It was truly amazing that when asked whether all 52 of the chips had to be used to explain the algorithm, these third graders said that it was not necessary to use all of them.

32 − 18 in Third Grade

This subtraction problem was given immediately after the preceding multiplication problem. After the child finished writing the answer, the interviewer put all the chips in front of the child saying, "I'd like you to explain with these chips how all this works that you wrote. Let's count out 32 chips

Table 2.7: Percentages in the Constructivist and Comparison Groups Explaining How They Got the Answer to 13 × 4

	Constructivist group (n = 13)	Comparison group (n = 39)	Difference	Significance (1-tailed)
Correct answer (52)	100	100	0	
Use of algorithm	0	97	97	.001
Adequate explanation of all the steps	92	5	87	.001
Interpreting the "1" of "13" as a one	0	87	87	.001

for this number that you had before taking 18 away [pointing to the 32 on the paper]."

Eighty-five percent of the constructivist group and 97% of the comparison group got the correct answer of 14 (see Table 2.8). Of those who produced the correct answer, 100% of the constructivist group and only 21% of the comparison group could explain how they got this answer. The difficulty of the comparison group was again due mostly to poor knowledge of place value. They used three chips (ooo) to explain how "borrowing one from three" and "taking one away from two" worked.

17 + 16 in Second Grade

In the spring of 2002, I (CK) gave this problem to two groups of low-SES second graders in California. The students in the constructivist group had been in the lowest math group at the beginning of first grade. They were then at such a low cognitive level that they could not conserve number or tell how many of four counters the interviewer had hidden. These children received constructivist instruction in first and second grades and were, of course, not taught any algorithms. This group of second graders was compared with a similar group in another school in the same low-SES neighborhood. In that school, math was taught with a state-approved textbook and workbook that included algorithms. To have a comparison group of about the same size, I asked for six low-performing students from 4 second-grade classes. (This request was made because

Table 2.8: Percentages in the Constructivist and Comparison Groups Explaining How They Got the Answer to 32 − 18

	Constructivist group (n = 13)	Comparison group (n = 39)	Difference	Significance (1-tailed)
Correct answer (14)	85	97	12	n.s.
Use of algorithm	0	100	100	.001
Adequate explanation of all the steps (percentages only of those who got the correct answer)	100	21	79	.001
Interpreting the tens as ones	0	87	87	.001

low-performing first graders usually become low-performing second graders.)

As can be seen in Table 2.9, significantly more students in the constructivist group wrote the correct answer of 33 (86% vs. 52%). When asked to explain their written procedures with a pile of 17 counters and another of 16, none of the group who had been taught algorithms could explain "carrying." By contrast, most (57%) of the constructivist group explained their procedures with tens and ones. (The others counted-on by ones.)

Second and Third Graders' Knowledge of the "1" in "17"

I showed a card on which "17" had been written to the same groups of second graders just discussed and asked each child to count out "this many" counters. I then circled the "7" and asked each child to show me with the chips "what this part means" and then circled the "1," asking "what this part means."

As can be seen in Table 2.10, 67% of the second graders in the constructivist group and only 4% of those in the comparison group showed 10 chips to explain the meaning of the "1" in "17." The difference between the two groups was significant at the .001 level.

Table 2.10 also shows third graders' knowledge of the "1" in "17." The third graders in the constructivist group had been in the lowest math group in first grade and had had 3 years of constructivist math without any algorithms. The third graders in the comparison group consisted of about four low-performing students from each of four classes, and these students had had 3 years of algorithms.

It can be seen in Table 2.10 that 100% of the constructivist group showed 10 counters to indicate the meaning of the "1" in "17." By contrast, only 35% of the comparison group indicated this knowledge ($p < .001$). The fact that 100% of the third graders in the constructivist group demonstrated knowledge of place value proves that even low-performing, low-SES children can learn place value if they are not taught any algorithms.

Table 2.9: Numbers and Percentages of Two Groups of Second Graders Solving 17 + 16

	Constructivist group (n = 21)	Comparison group (n = 23)	Difference	Significance (1-tailed)
Correct answer (33)	18 (86%)	12 (52%)	34%	.01
Use of algorithm	2 (10%)	13 (57%)	47%	.001
Perfect explanation with tens and ones	12 (57%)	0 (0%)	57%	.001

Table 2.10: Numbers and Percentages of Two Groups of Second and Third Graders Showing 10 Chips for the "1" in "17"

	Constructivist group [a]	Comparison group [b]	Difference	Significance (1-tailed)
Second grade	14 (67%)	1 (4%)	63%	.001
Third grade	12 (100%)	6 (35%)	65%	.001

[a] n = 21 for second grade; n = 12 for third grade

[b] n = 23 for second grade; n = 17 for third grade

CONCLUSION

Many more data on the harmful effects of teaching algorithms can be found in Chapter 10 of the present volume and Chapters 3 and 13 of *Young Children Continue to Reinvent Arithmetic, 3rd Grade* (Kamii, 1994). In conclusion, algorithms are harmful for two reasons: (1) They make children give up their own thinking, and (2) they "unteach" place value, thereby preventing children from developing number sense.

As stated earlier, children proceed from left to right when they do their own thinking in multidigit addition and subtraction. Because there is no compromise possible between going toward the right and going toward the left as the algorithms require, children have to give up their own thinking to obey the teacher. We have noted children's emotional flatness and lack of intuition when they get 29 or 900 for 7 + 52 + 186. These are symptoms of children who have given up their own thinking and are functioning like machines.

The algorithm of "carrying" serves to "unteach" place value by encouraging children to think about every digit as ones. In dealing with a problem like

$$\begin{array}{r} 48 \\ +\ 25 \\ \hline \end{array},$$

for example, children say, "Eight and five is thirteen. Put down the three; carry the one. One and four is five, plus two is seven." The algorithm is convenient for adults, who already know that the "4" and the "2" stand for 40 and 20. For second graders, who have a tendency to think about all the digits as ones, however, the algorithm serves to reinforce this error, as we saw in the examples given earlier.

There are a few second graders who are cognitively more advanced than the majority, know tens very well, and are therefore not harmed by the rules of "carrying." Adults certainly do not unlearn place value when they say, "One and four is five, plus two is seven," in the preceding situation. However, the great majority of second graders think "one," "four," and "two" when they say these words.

The harmful effects of algorithms have been suspected since the 1970s and 1980s (Carraher, Carraher, & Schliemann, 1985; Carraher & Schliemann, 1985; Jones, 1975; Plunkett, 1979). By the 1990s, an increasing number of educators were saying that the teaching of algorithms was harmful to children (Kamii, 1994; McNeal, 1995; Narode, Board, & Davenport, 1993; Pack, 1997; Parker, 1993; Richardson, 1996). However, authors of textbooks are still advocating the teaching of these rules, and "education reform" seems to mean only more testing, without any improvement in how we teach children. We conclude by wondering whether it will take 10 or 20 more years for algorithms to disappear from second-grade math books.

CHAPTER 3

The Importance of Social Interaction

EDUCATORS OFTEN SAY that peer interaction is important because children learn from each other. We agree that children learn many things from each other, but this is not our reason for advocating social interaction among peers in math classes. Logico-mathematical knowledge has its source inside each child and is elaborated through each child's own mental actions. In the logico-mathematical realm, therefore, other people are not the sources of knowledge. Rather, other people's ideas are important because they provide occasions for children to think critically about their own ideas in relation to other people's. For example, if one child says that 5 + 4 = 8, and another says that 5 + 4 = 9, this disagreement leads to critical thinking by both children, through the exchange of viewpoints. When children are convinced that someone else's idea makes better sense than theirs, they change their minds and correct themselves, *from the inside*.

Piaget (1980b) attributed great importance to social interaction. To him, such exchanges were indispensable, both for children's elaboration of logical thought and for adults' construction of sciences. As he put it:

> Certain educators say sometimes that my theory is only "cognitive" and that I neglected the importance of social aspects of the child's development. It is true that most of my publications have dealt with various aspects of cognitive development, particularly the development of operativity, but in my first works I emphasized the importance of interindividual exchanges sufficiently not to feel the need afterwards to return to it. In fact, it is clear that the confrontation of points of view is already indispensable in childhood for the elaboration of logical thought, and such confrontations become increasingly more important in the elaboration of sciences by adults. Without the diversity of theories and the constant search for going beyond the contradictions among them, scientific progress would not have been possible. (p. vii)

Piaget did not experimentally verify his theory about the importance of social interaction, but other researchers at the University of Geneva did. We have selected two studies by Doise and Mugny (1981/1984) as examples for this chapter. They demonstrate that even a 10-minute debate